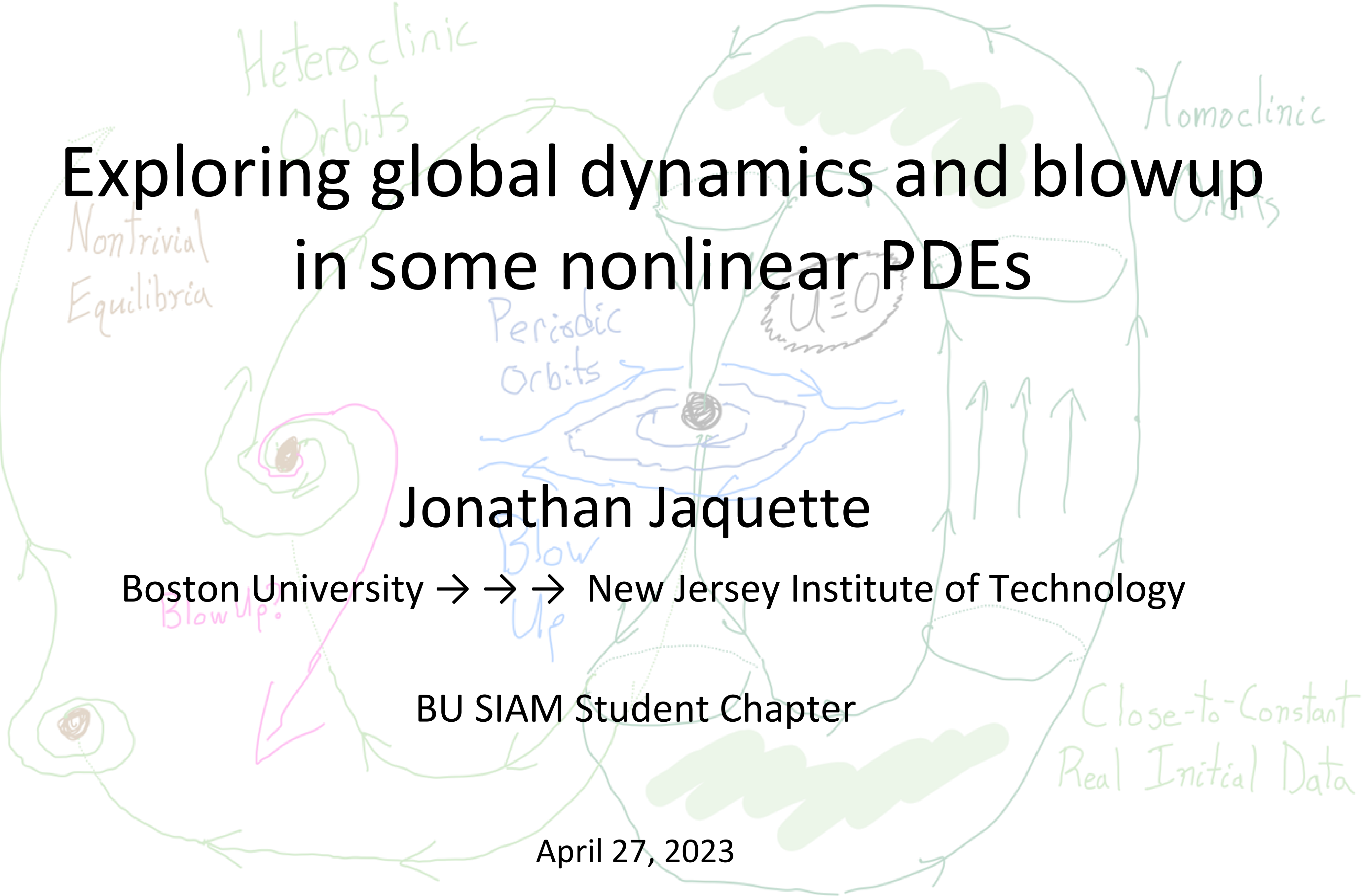
Exploring global dynamics and blowup in some nonlinear PDEs

Jonathan Jaquette

Boston University → → → New Jersey Institute of Technology

BU SIAM Student Chapter

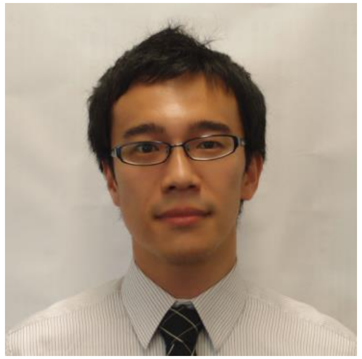
April 27, 2023



Collaborators

Jean-Philippe Lessard

McGill University



Akitoshi Takayasu

Tsukuba University

Outline

- **Part 1: Introduction**
- Part 2: What is a computer assisted proof?
- Part 3: A toy model for fluid dynamics
- Part 4: Global dynamics and blowup

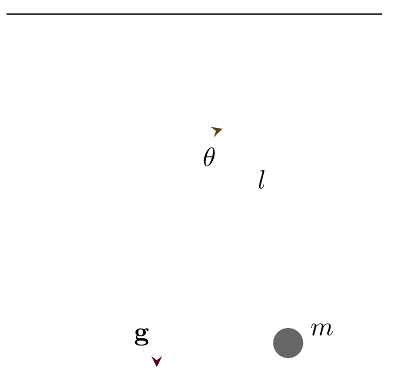


Image Credit: Wikipedia

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}\, t\right)$$

Natural World

Pendulum	Weather

?

Scientific Model

$\frac{d^2}{dt^2}\theta + \frac{g}{\ell}\sin\theta = 0$	Navier-Stokes; Rayleigh-Bénard convection

?

Toy Model

$\frac{d^2}{dt^2}\theta + \frac{g}{\ell}\theta = 0$	Lorenz System (ODE)

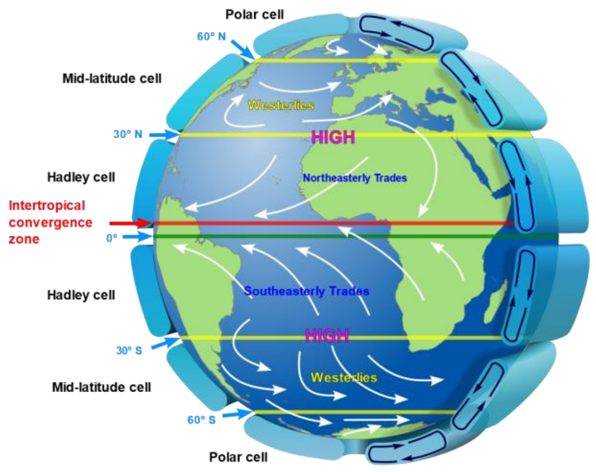
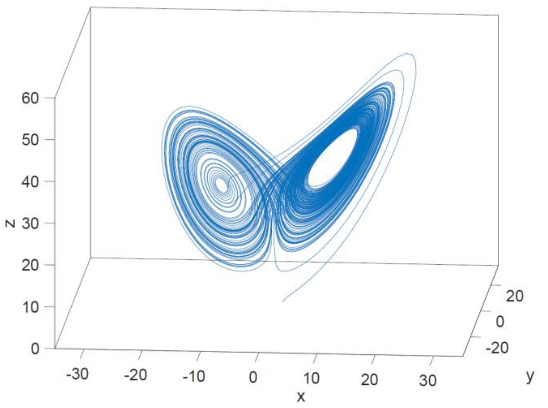
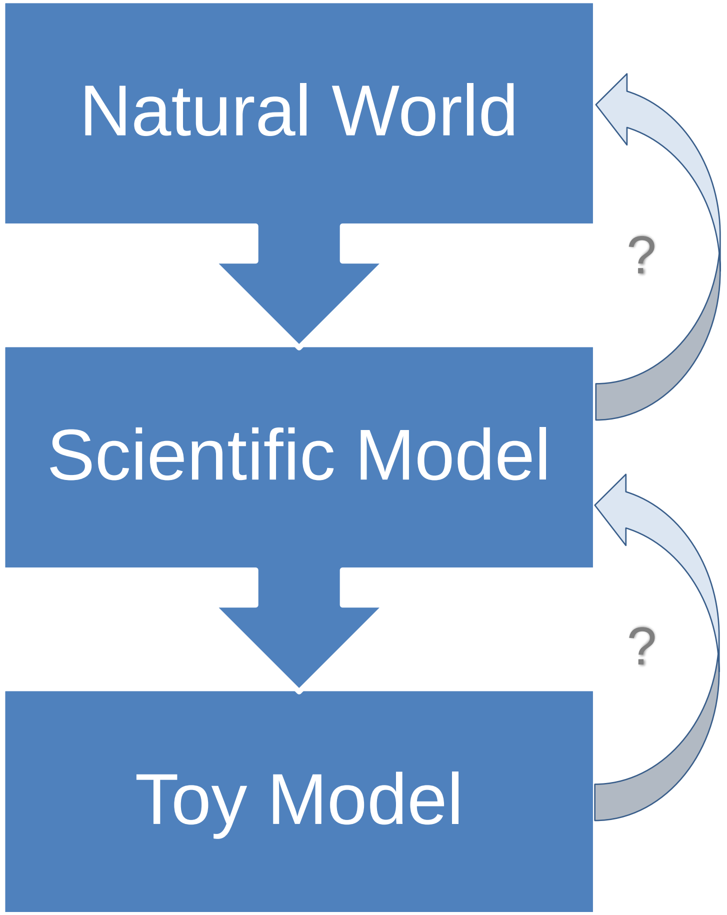


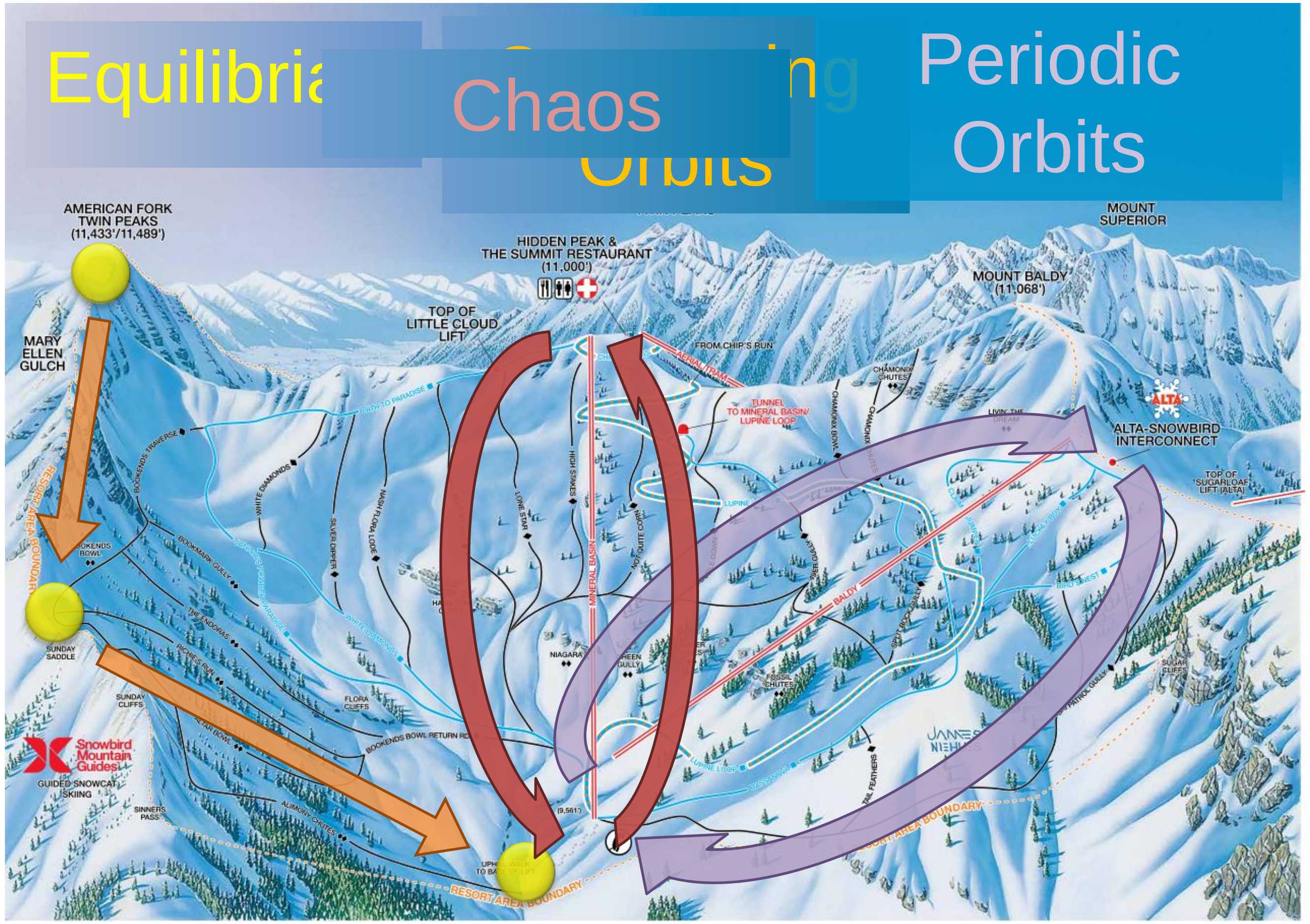
Image Credit: Wikipedia

Image Credit: Scientific Background on the Nobel Prize in Physics 2021; Weady et al. ‘18

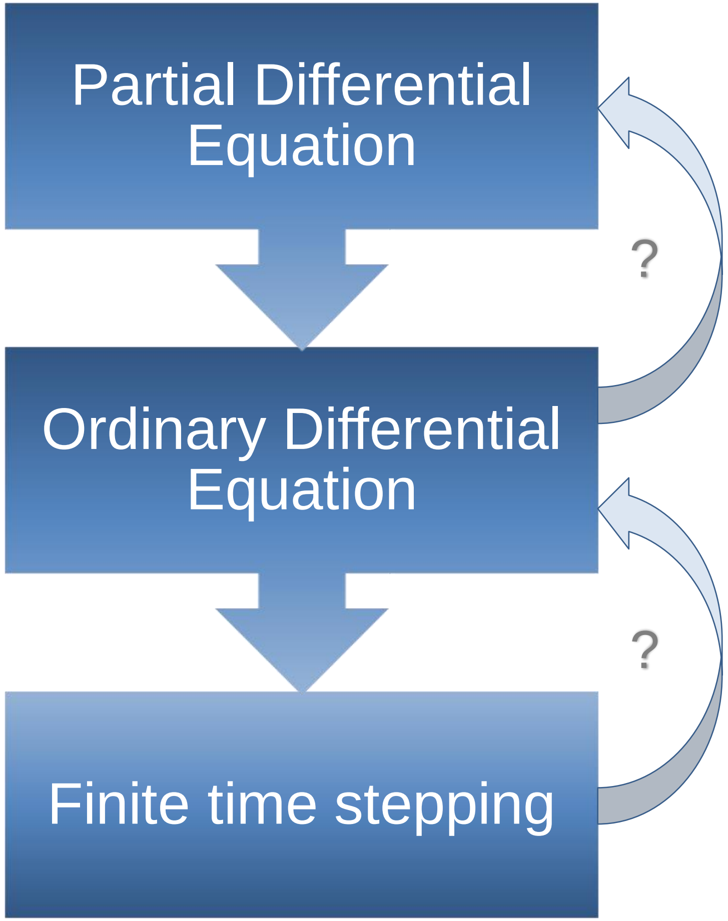


Which dynamical features are important?

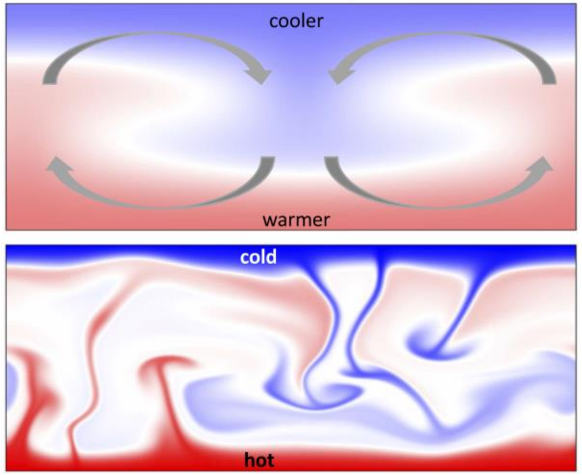




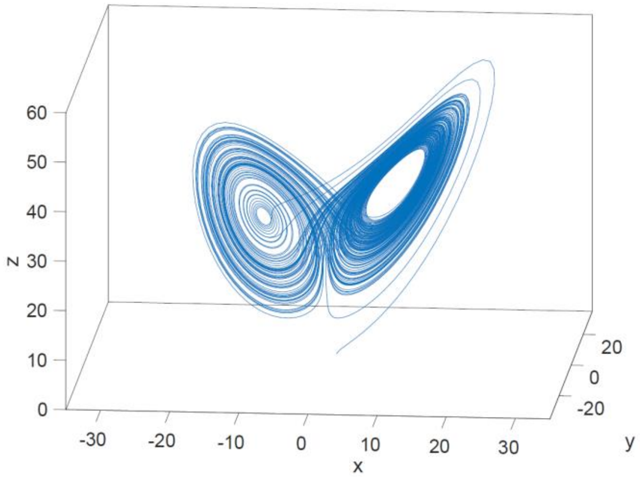
Which dynamical features persist?



- Numerical approximations converge in the limit
  - How accurate is a particular computation?

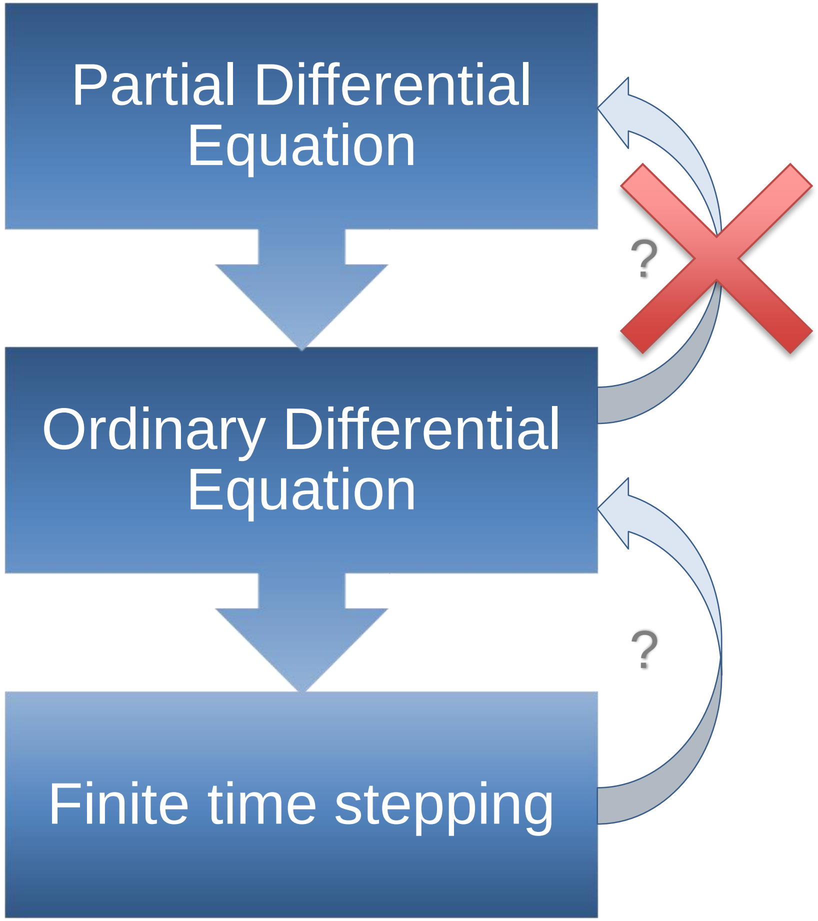


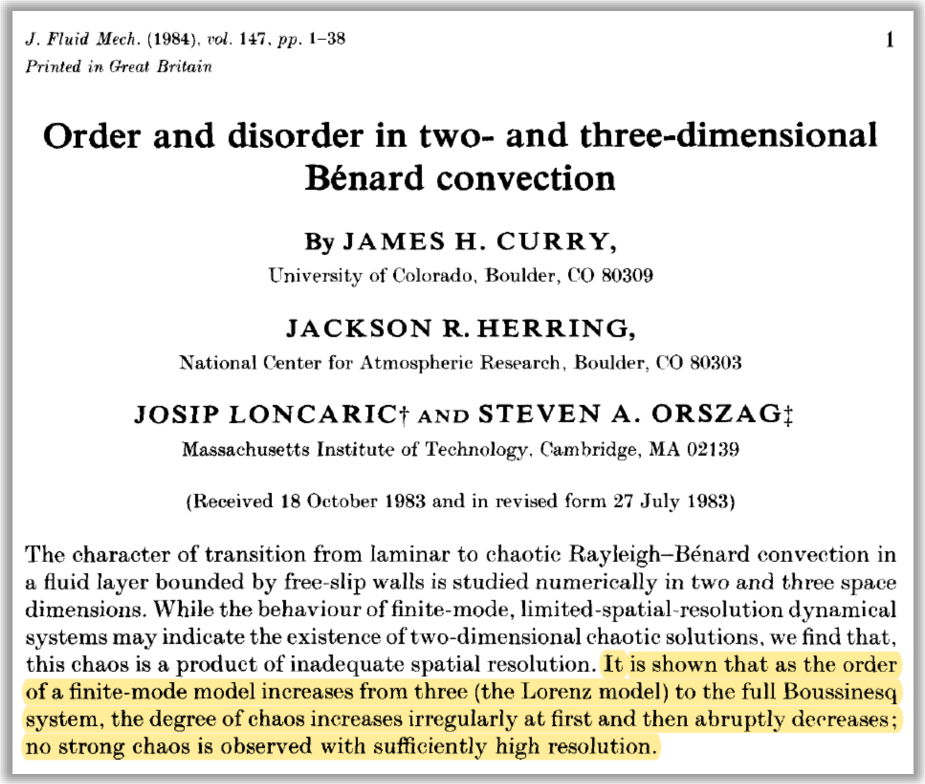
Temperature field in 2D Rayleigh-Bénard convection simulations. Image Credit: Doering 2020



The Lorenz attractor, a 3-mode approx. of Rayleigh-Bénard convection. Image Credit: Weady et al. '18

Which dynamical features persist?





J. Fluid Mech. (1984), *vol.* 147, *pp.* 1–38 1
Printed in Great Britain

Order and disorder in two- and three-dimensional Bénard convection

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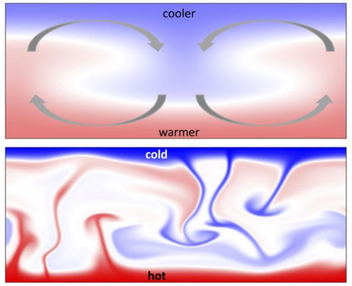
(Received 18 October 1983 and in revised form 27 July 1983)

The character of transition from laminar to chaotic Rayleigh–Bénard convection in a fluid layer bounded by free-slip walls is studied numerically in two and three space dimensions. While the behaviour of finite-mode, limited-spatial-resolution dynamical systems may indicate the existence of two-dimensional chaotic solutions, we find that, this chaos is a product of inadequate spatial resolution. It is shown that as the order of a finite-mode model increases from three (the Lorenz model) to the full Boussinesq system, the degree of chaos increases irregularly at first and then abruptly decreases; no strong chaos is observed with sufficiently high resolution.



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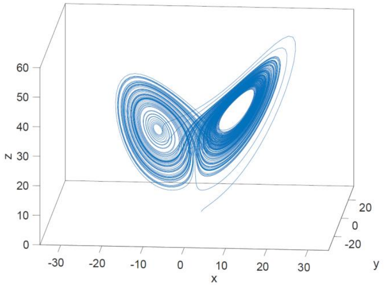

The Lorenz attractor, a 3-mode approx. of Rayleigh-Bénard convection. Image Credit: Weady et al. '18

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What is a Computer Assisted Proof?

My Definition: *A proof involving computations.*

e.g. 109 is prime; $9 < \pi^2 < 10$

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Sieve of Eratosthenes

`input`: integer n

`output`: primes between 2 & n

$S := \{2,3,4 \ldots, n\}$
$p := 2$
`while` $p \leq \sqrt{n}$
  remove $2p, 3p, 4p, \ldots$ from S
  $p \leftarrow$ smallest $x \in S, x > p$
`return` S

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Prime numbers

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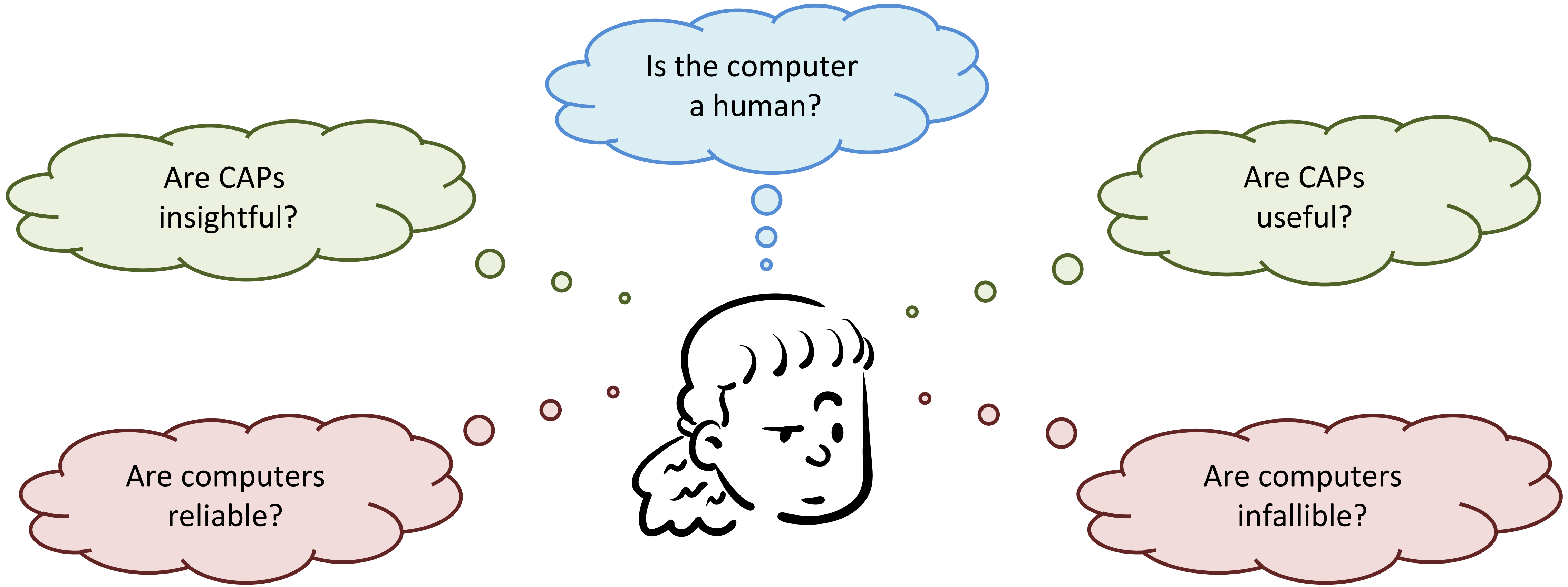
e.g. 109 is prime; $9 < \pi^2 < 10$

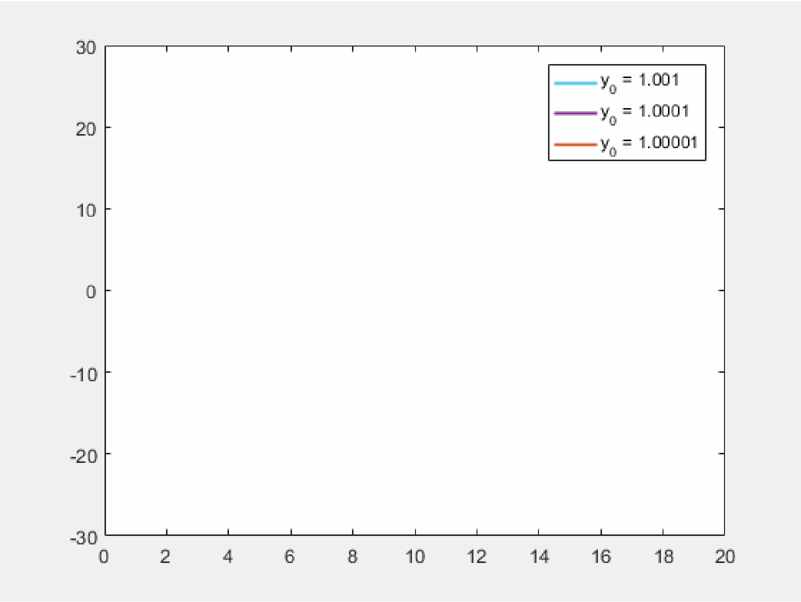


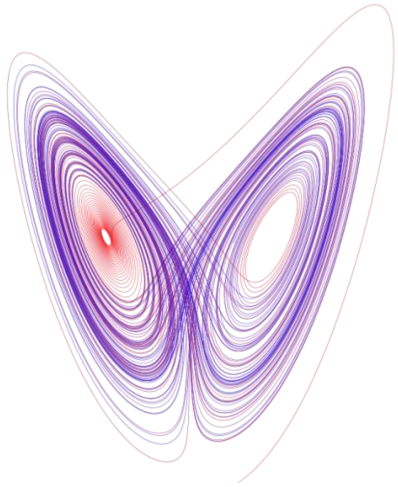
Numerics gone awry

- In 1963 Edward Lorenz was studying following model for atmospheric convection

$$x' = \sigma(y - x)$$
$$y' = x(\rho - z) - y$$
$$z' = xy - \beta z$$

- Origin of the term 'Butterfly Effect'
  - Sensitive dependance to initial conditions
  - Under modern conventions, Ellen Fetter would have been a co-author
    - https://www.quantamagazine.org/the-hidden-heroines-of-chaos-20190520/





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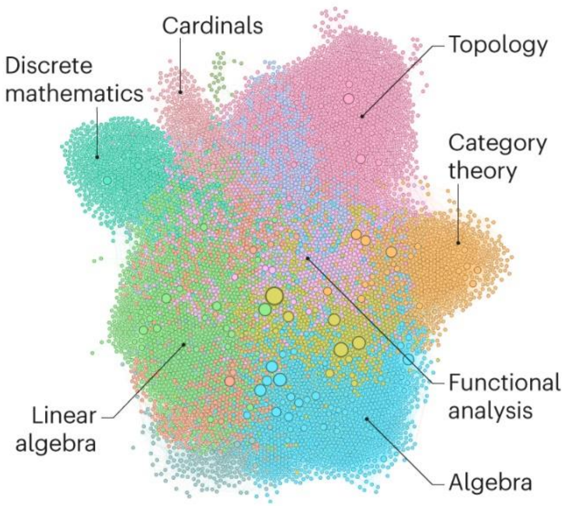
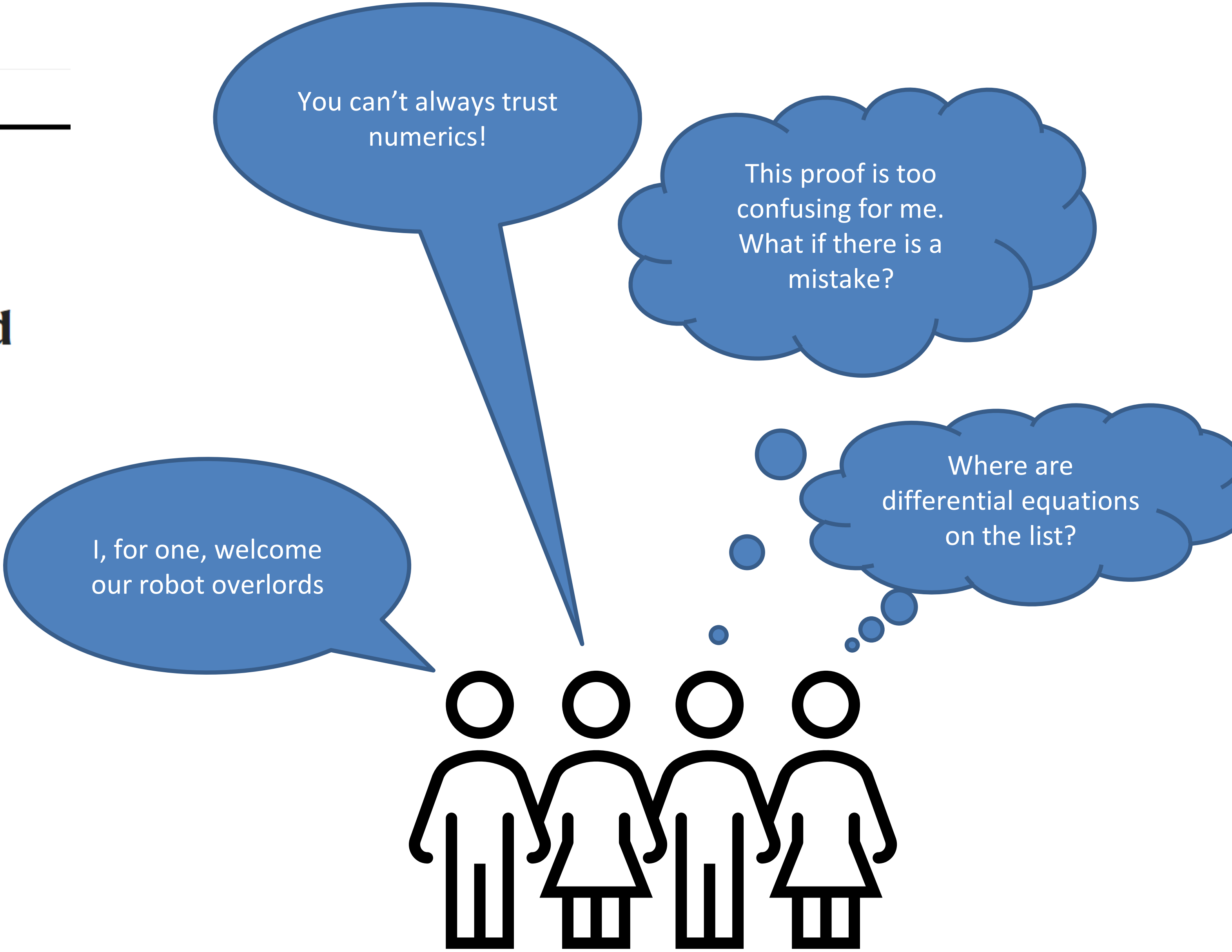
NEWS | 18 June 2021

Mathematicians welcome computer-assisted proof in 'grand unification' theory

Proof-assistant software handles an abstract concept at the cutting edge of research, revealing a bigger role for software in mathematics.

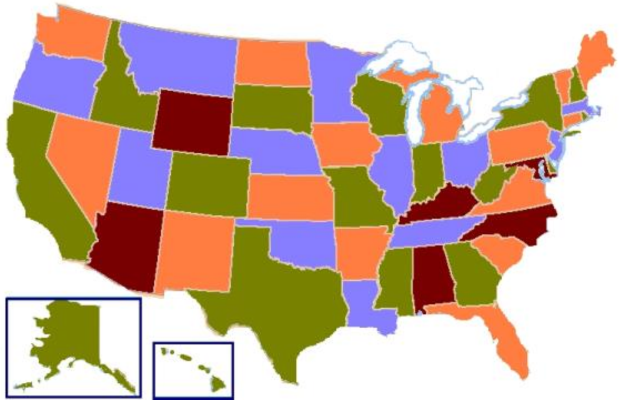
Davide Castelvecchi

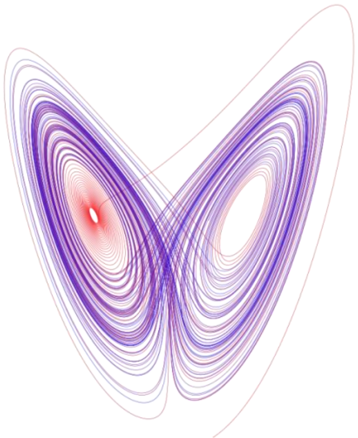


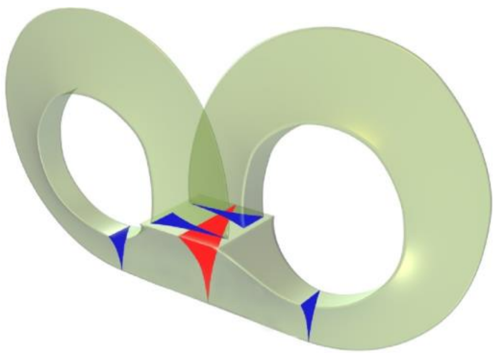


Famous Computer Assisted Proofs

- Four Color Theorem
  - How many colors are needed so adjacent countries have different colors on a map? (1852)
  - C.A.P. by Appel & Haken (1976)
    - Reduced to ~1,500 possible counter-examples
- The Lorenz system
  - Standard model of chaos
    - C.A.P. by Mischaikow & Mrozek (1995)
  - Smale's 14^{th} problem for the 21^{st} century
    - Does the Lorenz attractor match the geometric model?
    - C.A.P. by Tucker (2002)







Easy Part: living with rounding error

- Computers have finite memory
- **Interval arithmetic**
  - Define real intervals as
    $\mathbb{IR} = \{[a, b] \subseteq \mathbb{R} : a \leq b\}$
  - Define operations $\star \in \{+, -, \times, /\}$ as
    $A \star B = \{\alpha \star \beta : \alpha \in A, \beta \in B\}$

Examples

$$[1,2] + [3,4] = [4,6]$$

$$[1,2] - [3,4] = [-3, -1]$$

$$[1]/[3] \in [\,0.33, 0.34]$$

$$\pi \in [\,3.1, 3.2]$$

$$\pi^2 \in [\,9.61, 10.24]$$

$f(x) = x^5 - x + 1$

- **Goal**: Solve $f(x) = 0$
- Consider interval $I = [-2,-1]$

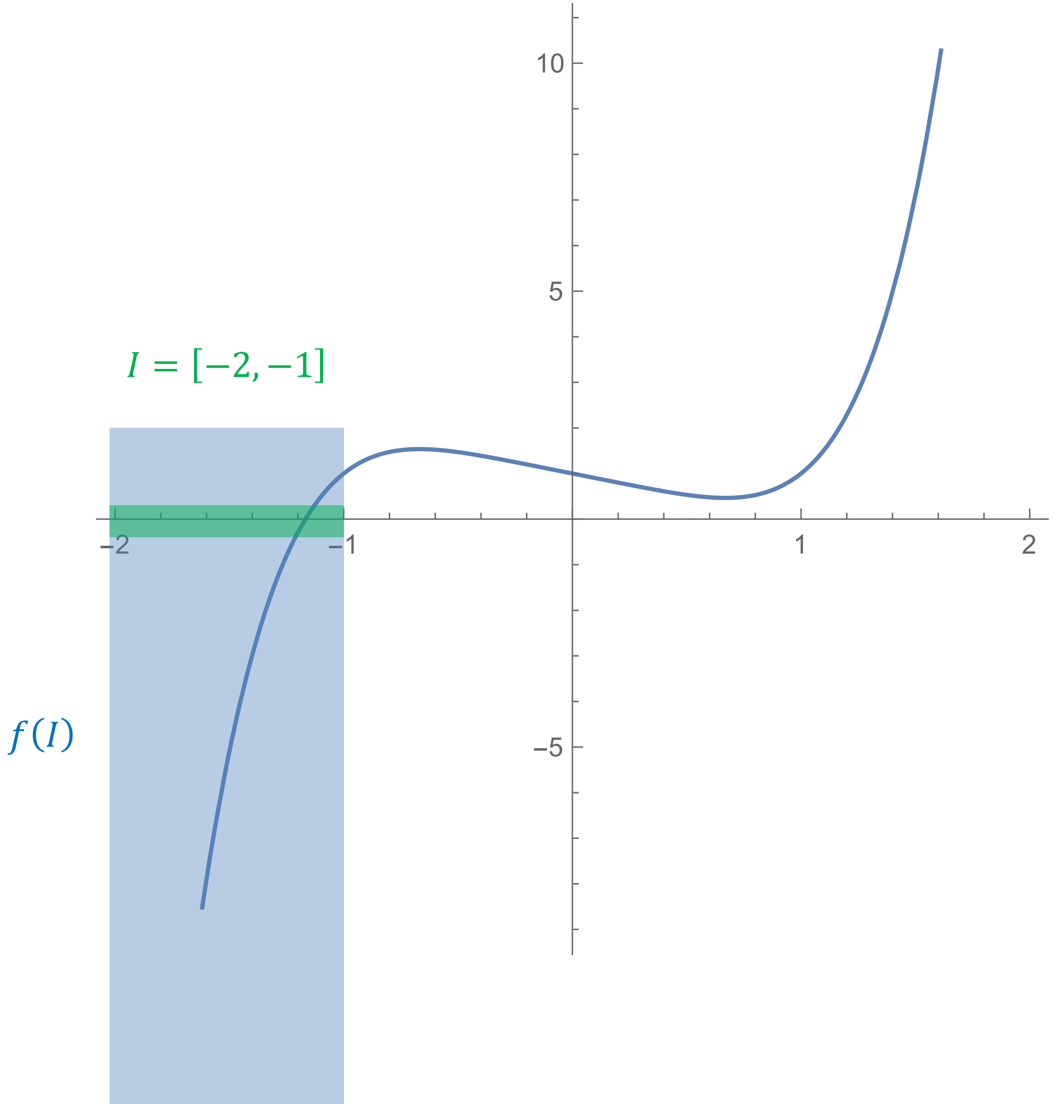
  $f(I) = [-2,-1]^5 - [-2,-1] + [1,1]$

  $= [-32,-1] + [2,3]$

  $= [-30,2]$

Theorem (with computer assisted proof): There exists a unique $\tilde{x} \in [-2,-1]$ such that $f(\tilde{x}) = 0$.

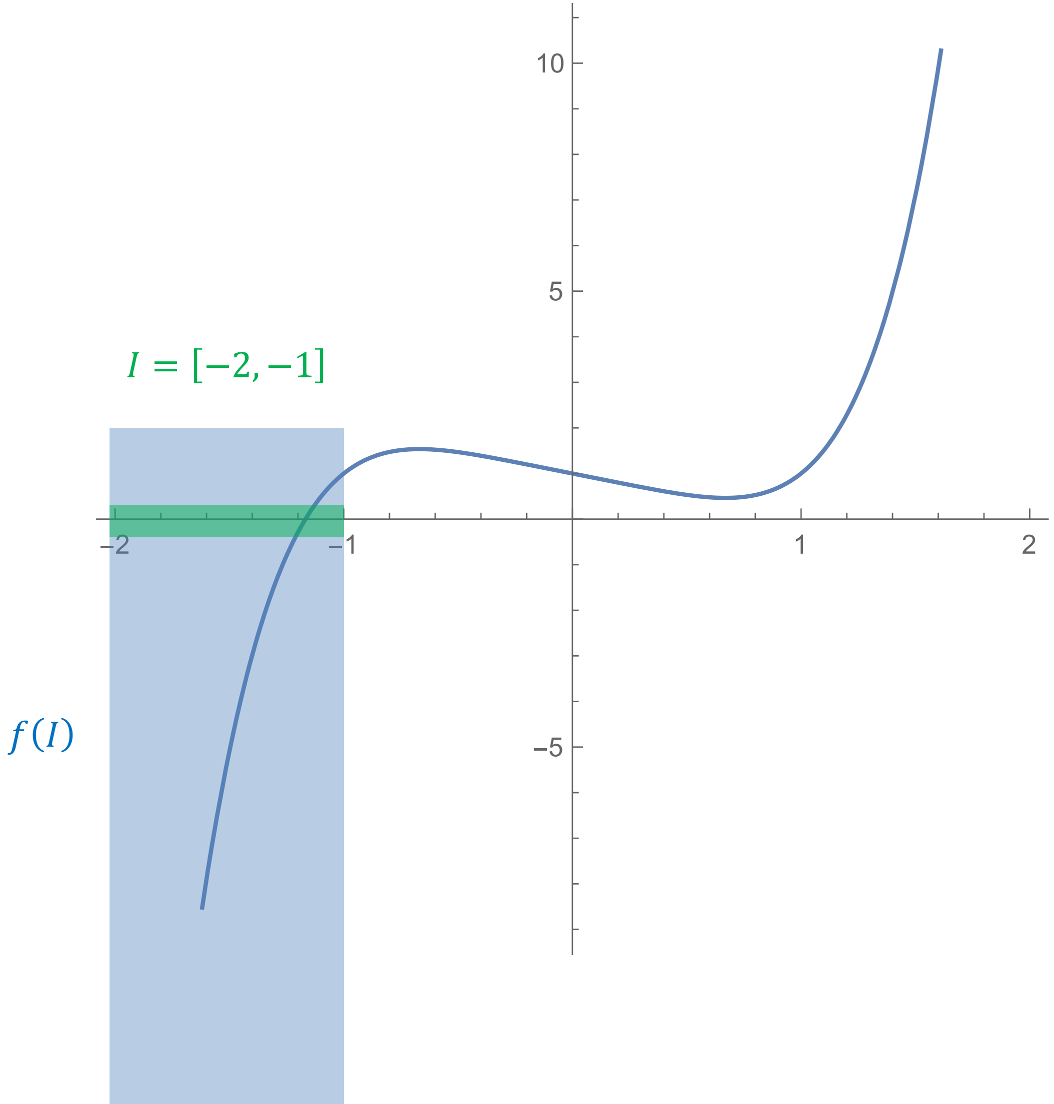
- Use intermediate value theorem to show that a solution exists
  - $f(-2) = -29 < 0$
  - $f(-1) = +1 > 0$
- Uniqueness
  - $f'(I) = [\,4,\ 79\,] > 0$



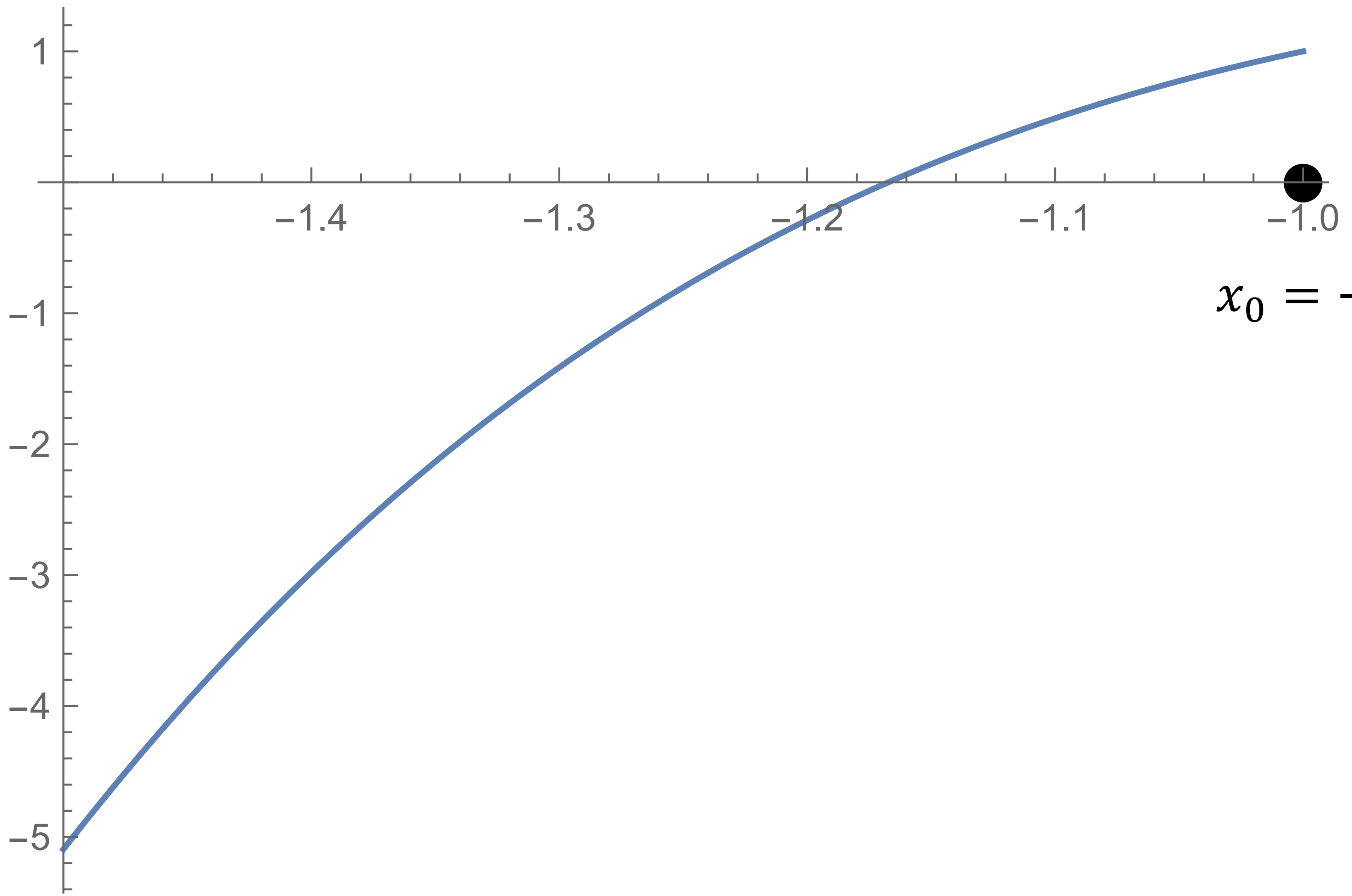
$$f(x) = x^5 - x + 1$$

Theorem (with computer assisted proof): There exists a unique $\tilde{x} \in [-2, -1]$ such that $f(\tilde{x}) = 0$.

Corollary: There exists a unique $\tilde{x} \in \mathbb{R}$ such that $f(\tilde{x}) = 0$.
Proof: Divide and conquer

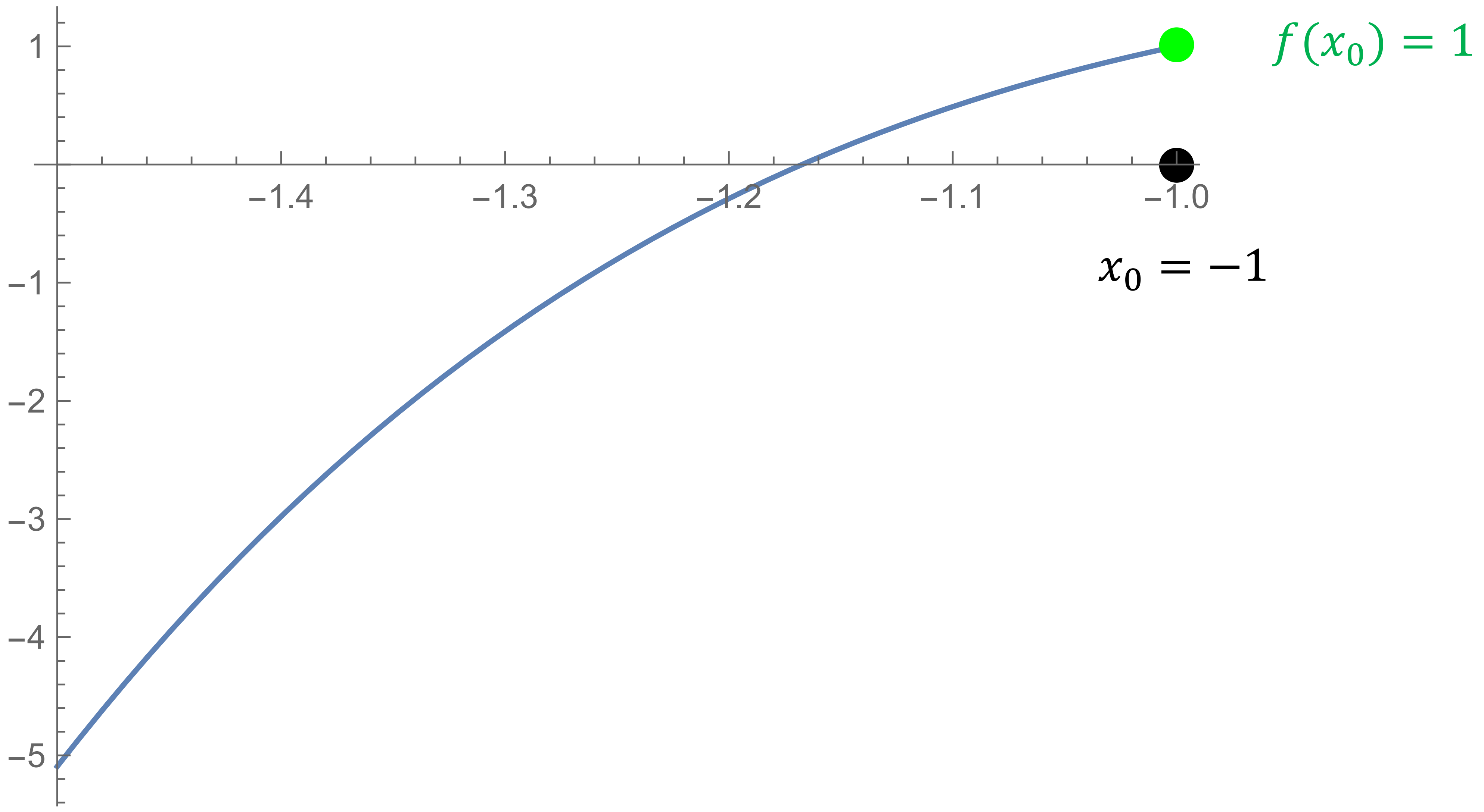


Newton's method: $x_{n+1} = x_n - f'(x_n)^{-1} f(x_n)$



$x_0 = -1$

Newton's method: $x_{n+1} = x_n - f'(x_n)^{-1} f(x_n)$



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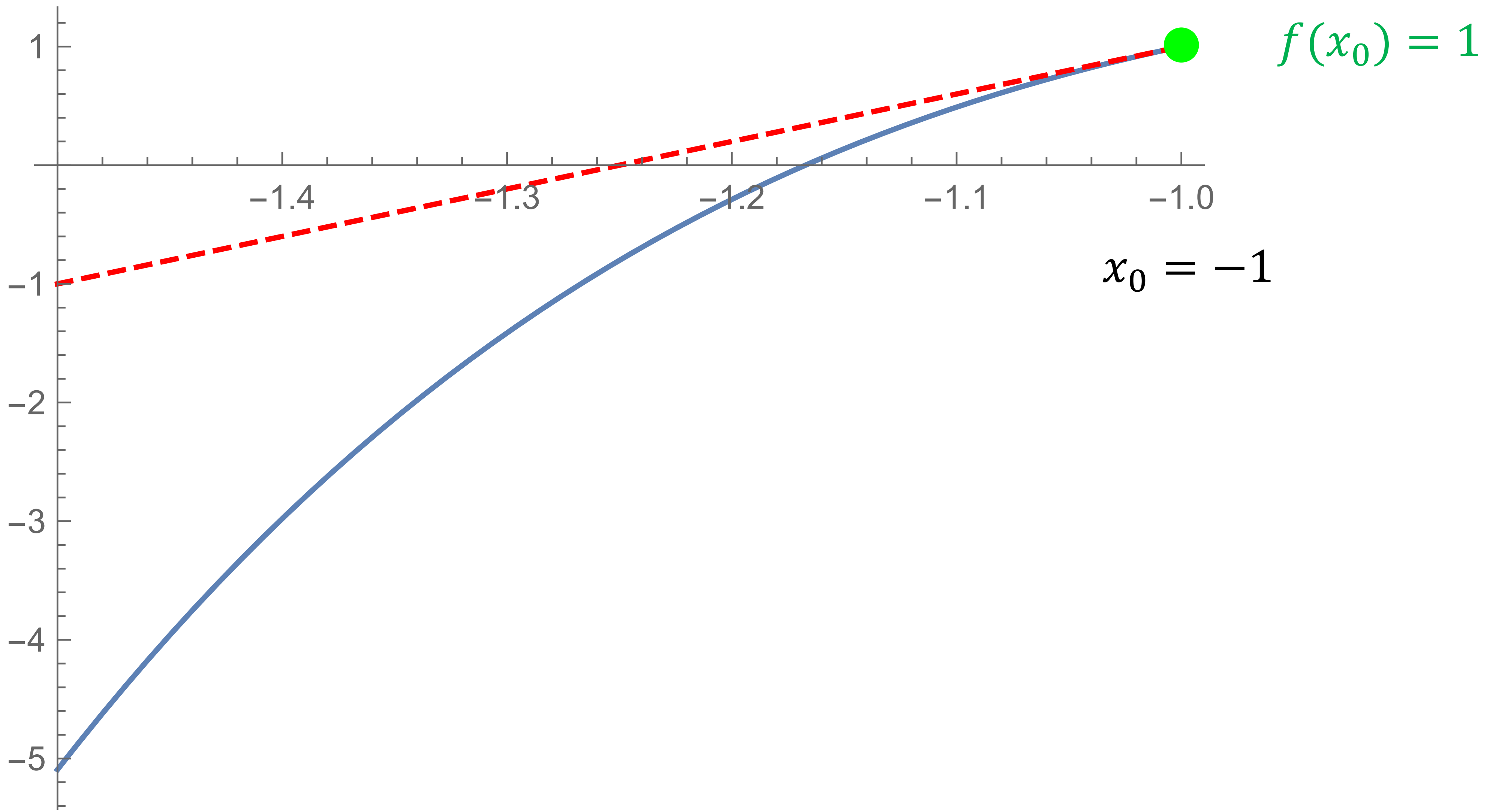

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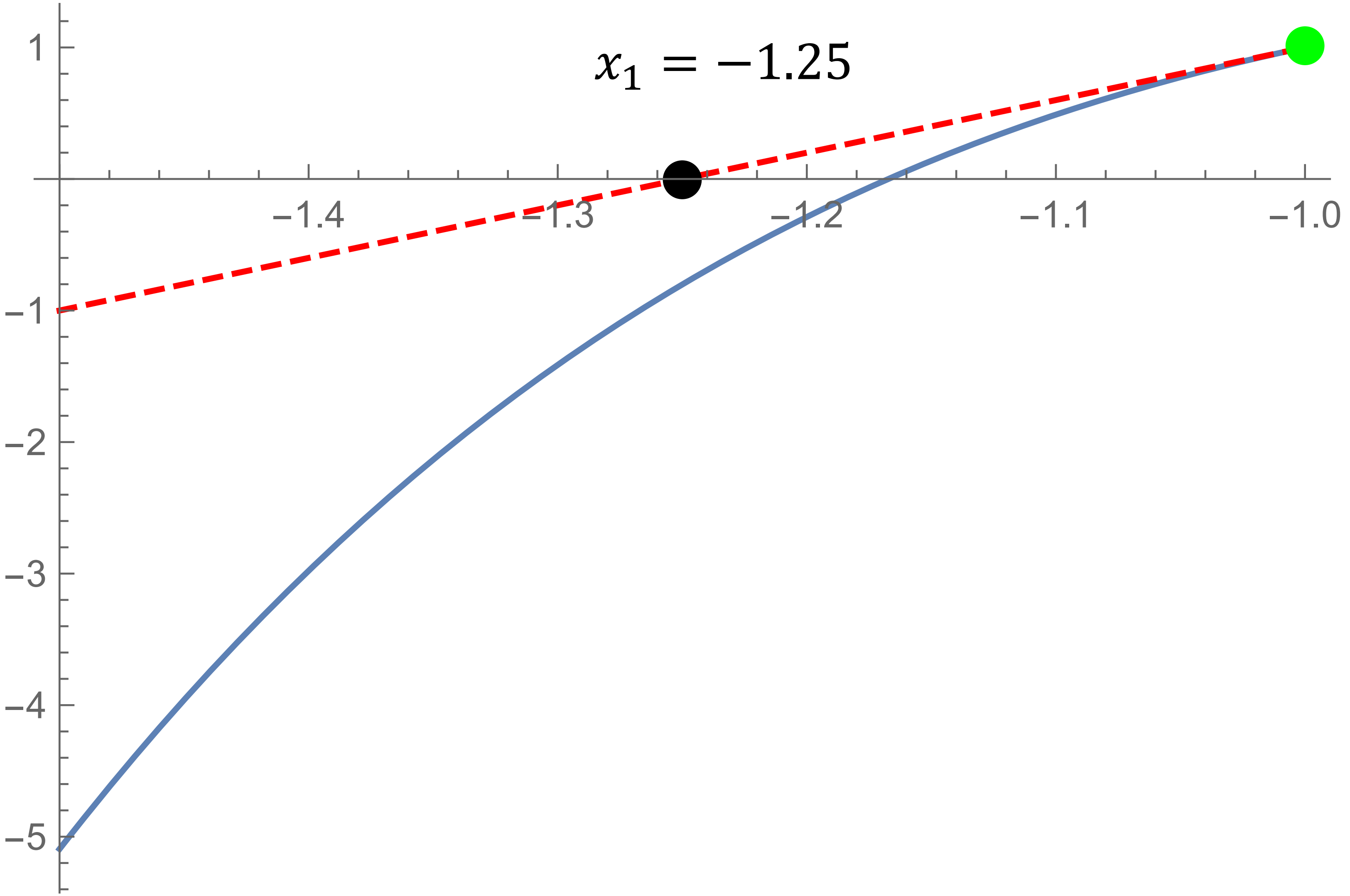

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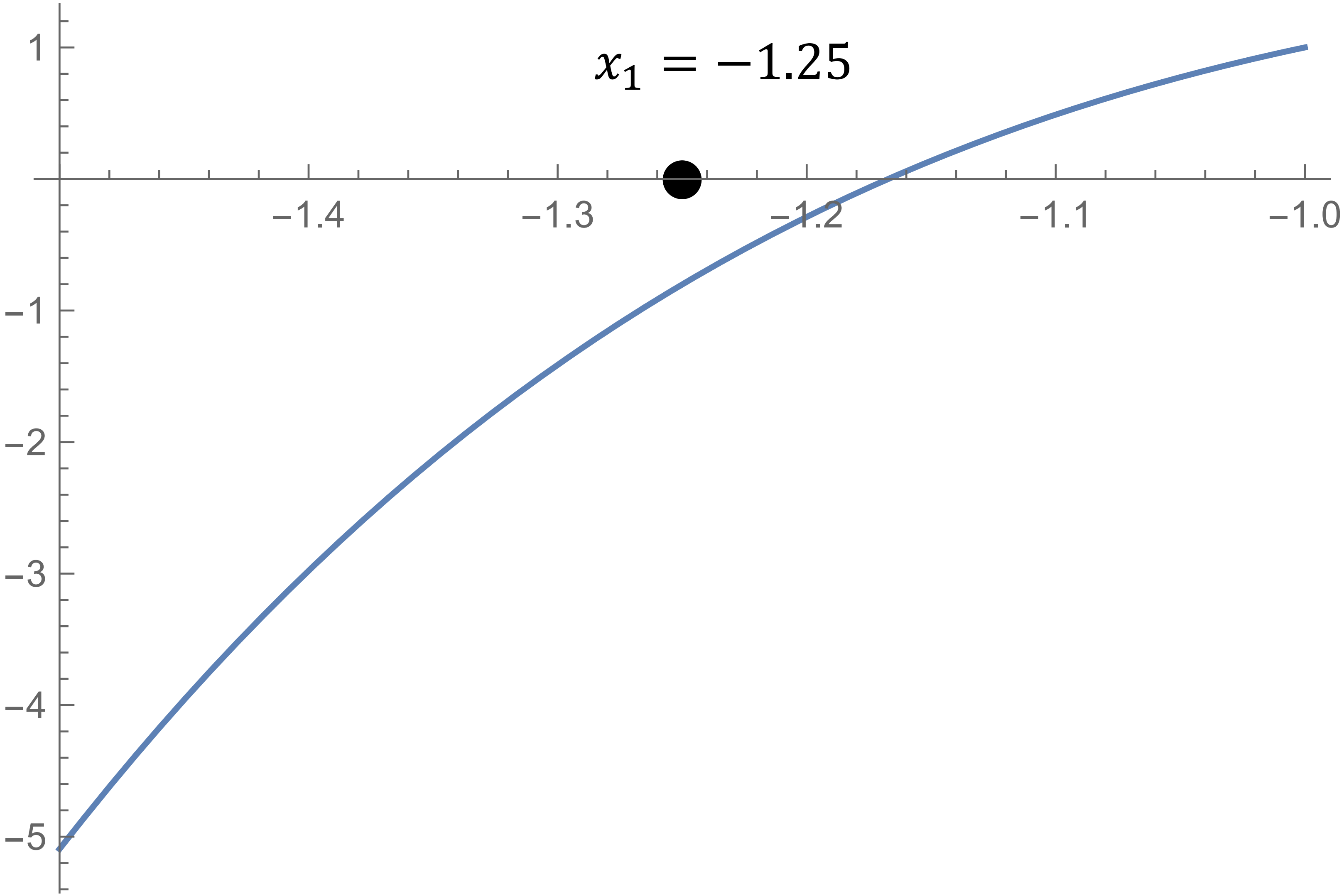

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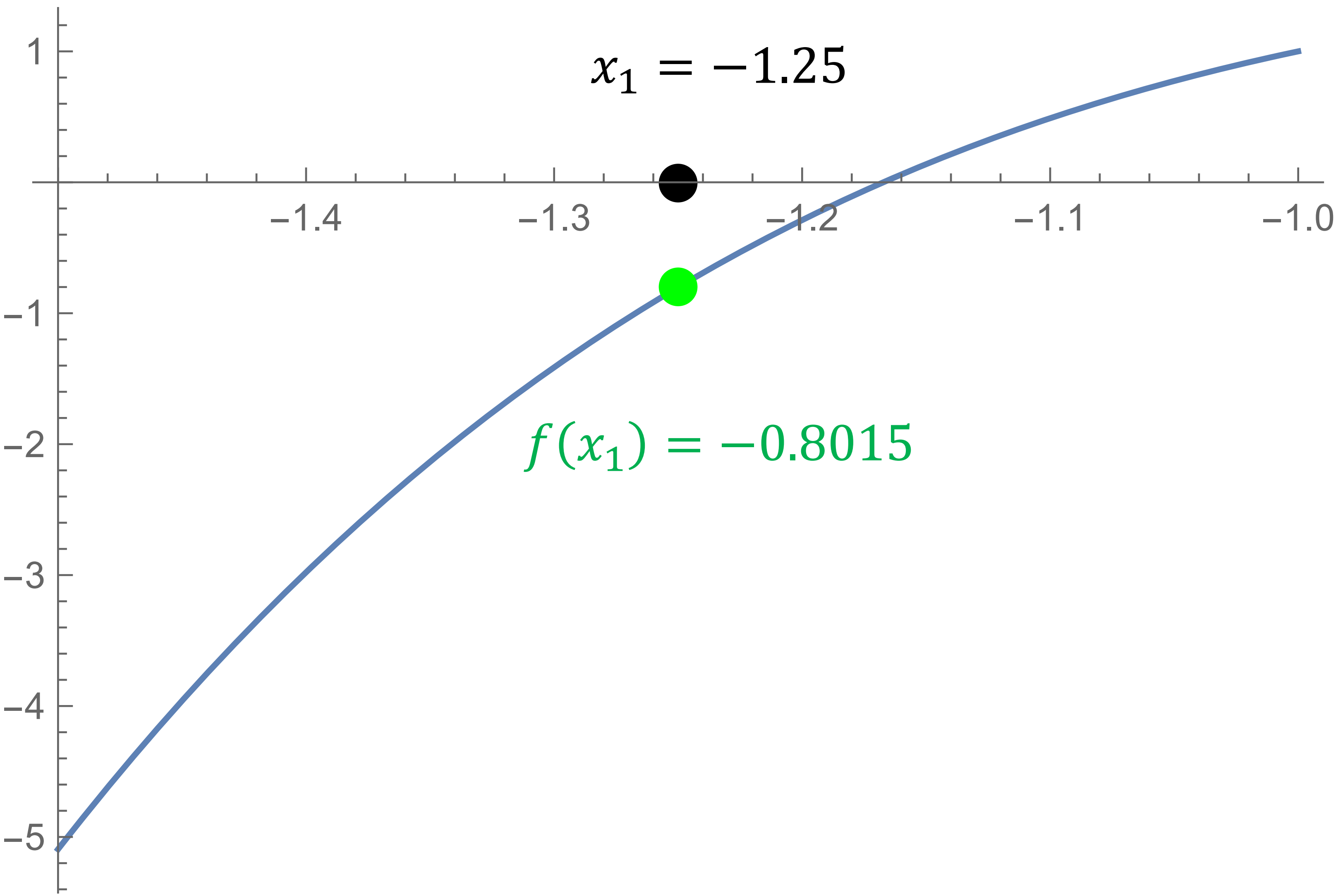

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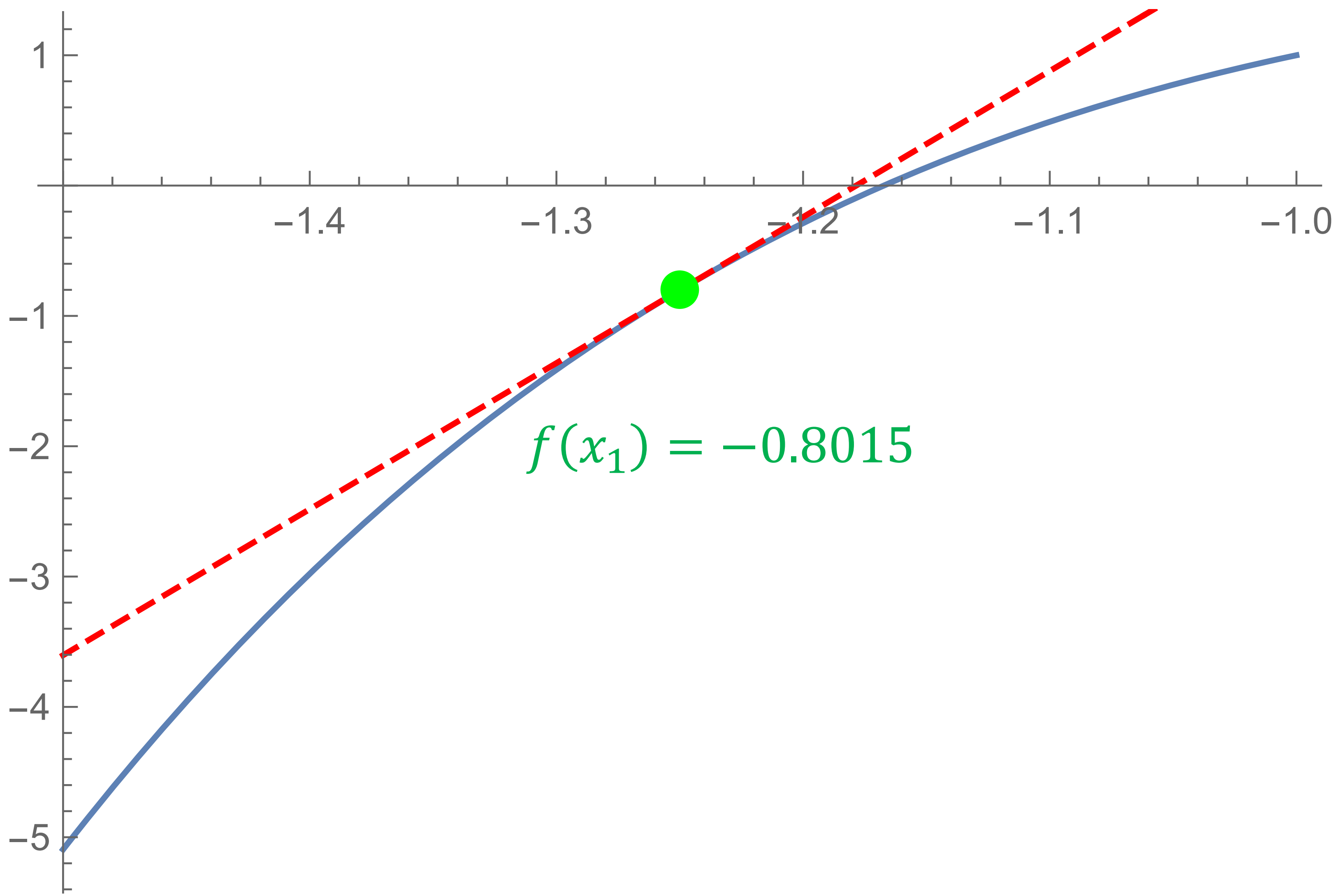

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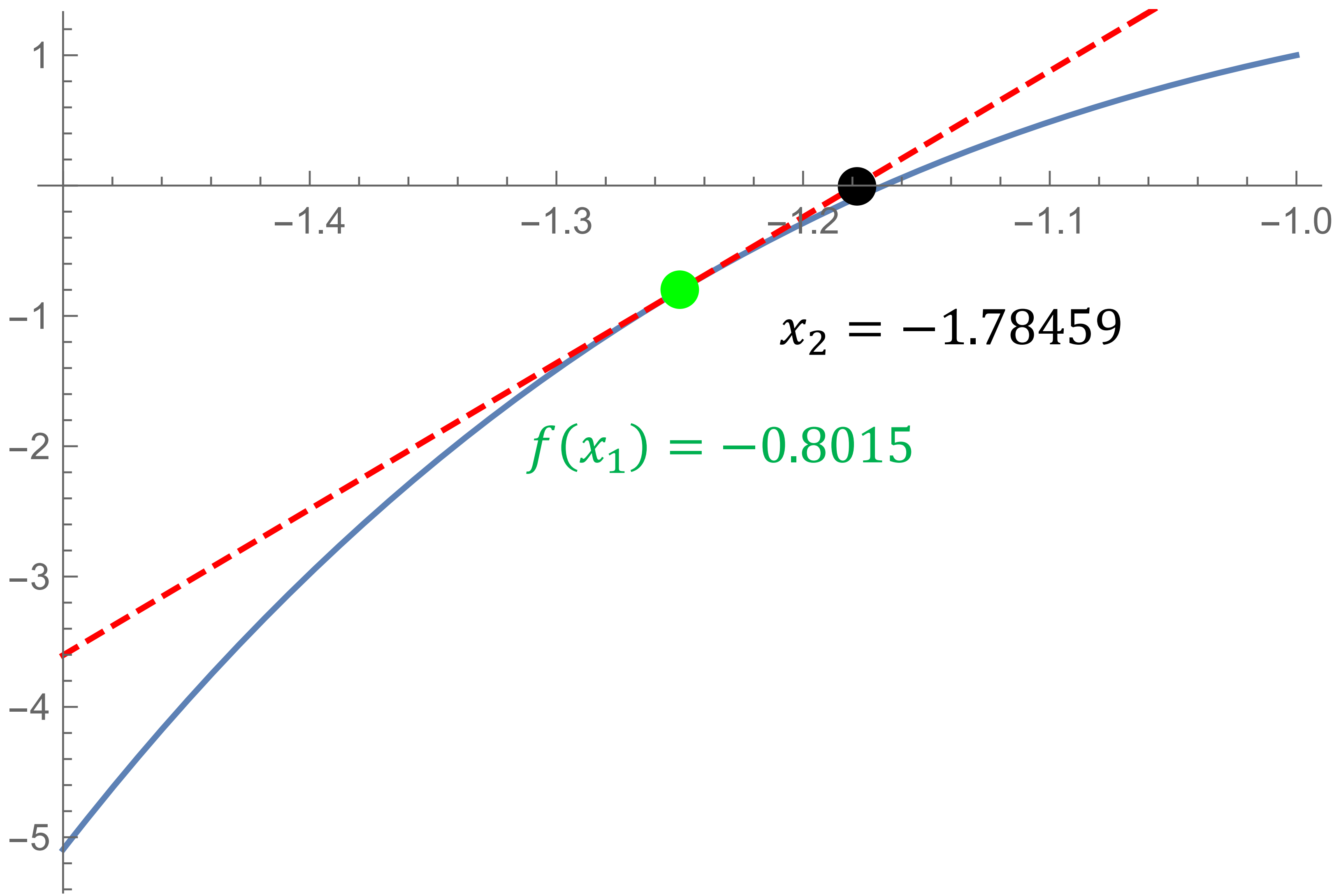

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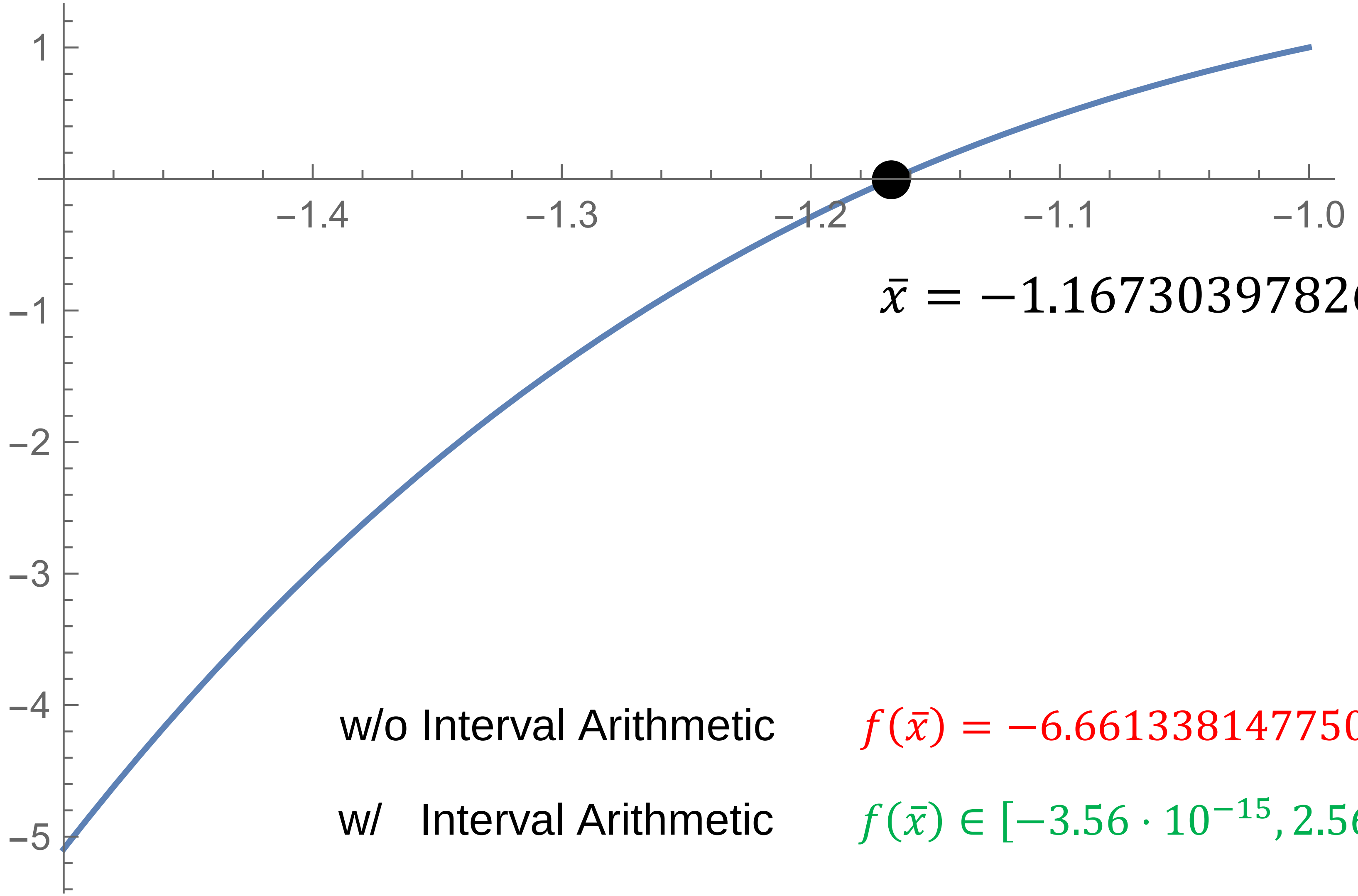


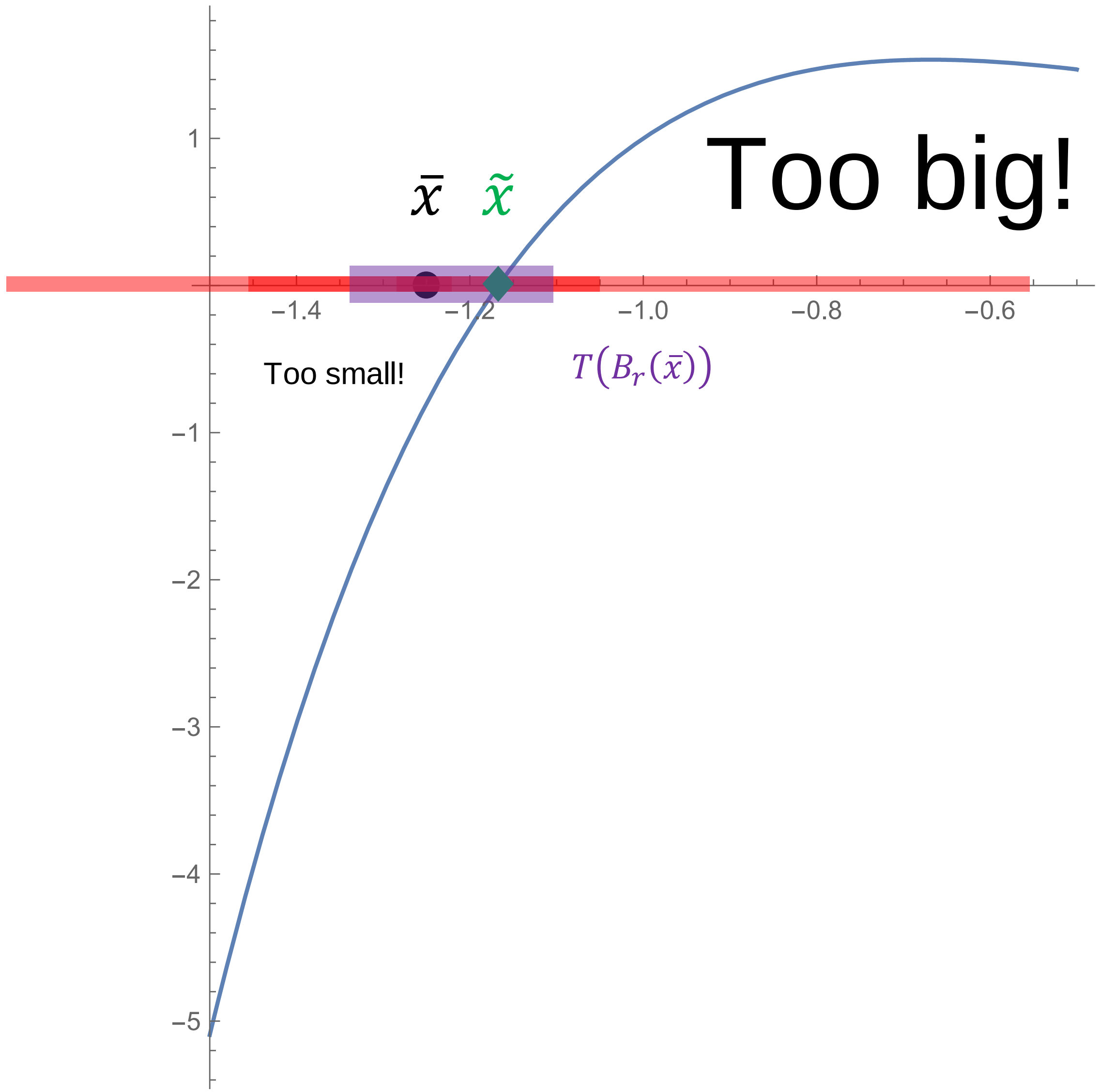
$\bar{x} = -1.1673039782614187$

w/o Interval Arithmetic $f(\bar{x}) = -6.661338147750939 \cdot 10^{-16}$

w/ Interval Arithmetic $f(\bar{x}) \in [-3.56 \cdot 10^{-15}, 2.56 \cdot 10^{-15}]$

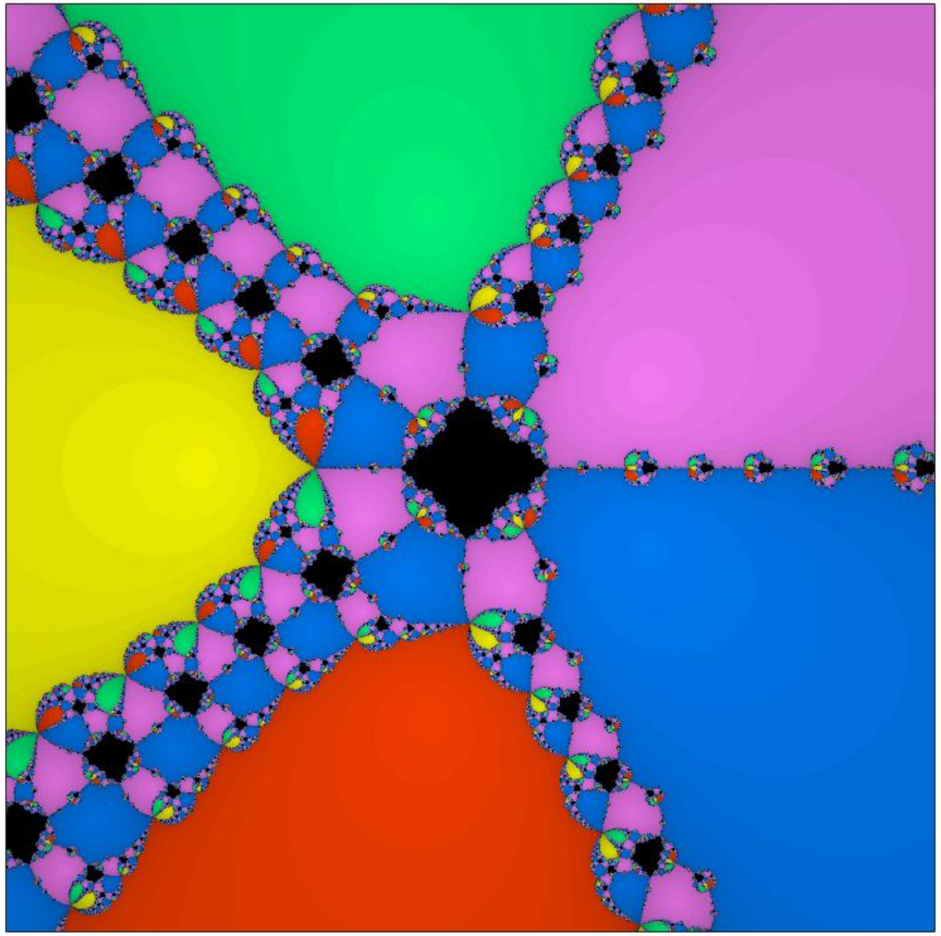
How to prove $f(x) = 0$

- **Define:** Newton map
$$T(x) = x - f'(x)^{-1} f(x)$$
- **Define:** $B_r(\bar{x})$, a closed ball about $\bar{x}$ of radius r
- **Goal:** Show that T is a **contraction mapping:**
  - T maps $B_r(\bar{x})$ into itself
  - points get closer together
- **Th'm:** If T is a contraction, then $B_r(\bar{x})$ contains a unique fixed point $\tilde{x}$
$$T(\tilde{x}) = \tilde{x} \quad \Longleftrightarrow \quad f(\tilde{x}) = 0$$
- How to choose the right value of $\boldsymbol{r}$?

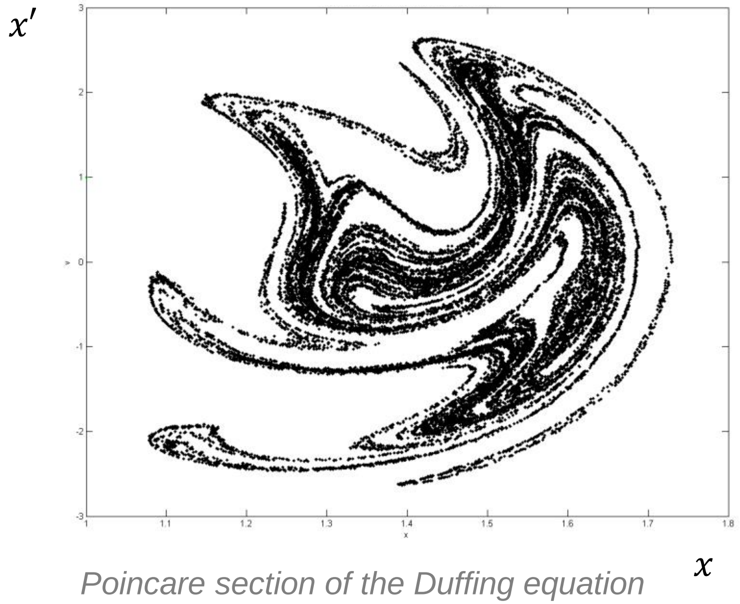


Newton's method in higher dimensions

- There are complex roots to
  $$f(x) = x^5 - x + 1$$
- If $f : \mathbb{R}^n \to \mathbb{R}^n$ define Newton map
  $$T(x) = x - Df(x)^{-1} f(x)$$
- Newton Fractal
  - The colors represent basins of attraction
  - Black means Newton's method did not converge



Hard Part: ∞-dimensional problems



Poincare section of the Duffing equation with $\alpha = 1, \beta = 5, \epsilon = 0.02, \gamma = 8, \omega = 0.5.$
Image Credit: Wikipedia

Consider the Duffing equation for a **damped driven oscillator**

$$x'' + \epsilon\, x' + \alpha x + \beta x^3 = \gamma \cos \omega t$$

To look for 2π periodic solution ($\omega = 1$), expand $x(t)$ as a Fourier series

$$x(t) = \sum_{k\in\mathbb{Z}} a_k\, e^{ikt}$$

where $a_{-k} = (a_k)^*$. Inserting into the ODE, we obtain

$$\sum_{k\in\mathbb{Z}} (-k^2 + i\epsilon k + \alpha) a_k e^{ikt} + \beta \left(\sum_{k\in\mathbb{Z}} a_k e^{ikt} \right)^3 = \gamma \left(e^{it} + e^{-it}\right)/2$$

Matching the e^{ikt} terms, we obtain equations $\forall k \in \mathbb{Z}$

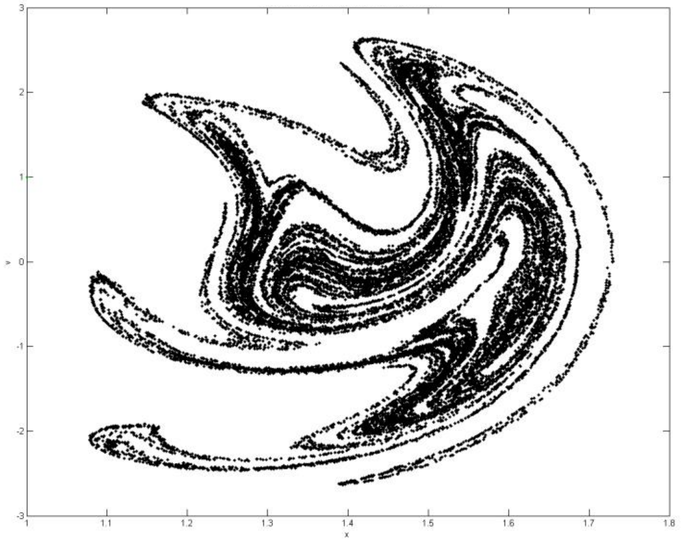
$$0 = (-k^2 + i\epsilon k + \alpha) a_k + \beta \sum_{\substack{k_1+k_2+k_3=k;\\ k_1,k_2,k_3\in\mathbb{Z}}} a_{k_1} a_{k_2} a_{k_3} - \gamma \delta_{1,k}/2$$

$$\stackrel{\text{def}}{=} f_k(a)$$

$$f_k(a) \approx (-k^2 + i\epsilon k) a_k + \mathcal{O}\left(\|a\|_{\ell^1}^3\right)$$

$$a_k = \mathcal{O}\left(\frac{1}{k^2}\right)$$

Hard Part: ∞-dimensional problems



Poincare section of the Duffing equation with $\alpha = 1, \beta = 5, \epsilon = 0.02, \gamma = 8, \omega = 0.5$.
Image Credit: Wikipedia

$$f_k(a) \approx (-k^2 + i\epsilon k)a_k + \mathcal{O}\left(\|a\|_{\ell^1}^3\right)$$

$$a_k = \mathcal{O}\left(\frac{1}{k^2}\right)$$

- **Theorem:** A periodic orbit $x(t)$ is equivalent to a solution $f(a) = 0$
- **Define:** Galerkin truncation $f^N: \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$
  - Find approximate solution $\hat{a} \in \mathbb{R}^{2N+1}$ such that $f^N(\hat{a}) \approx 0$
- **Define:** Quasi-Newton map on the whole ∞-dimensional space
  $$T(a) = a - Af(a),$$
  $$A \approx Df(\hat{a})^{-1}$$
- **Goal:** Show that T is a **contraction mapping***

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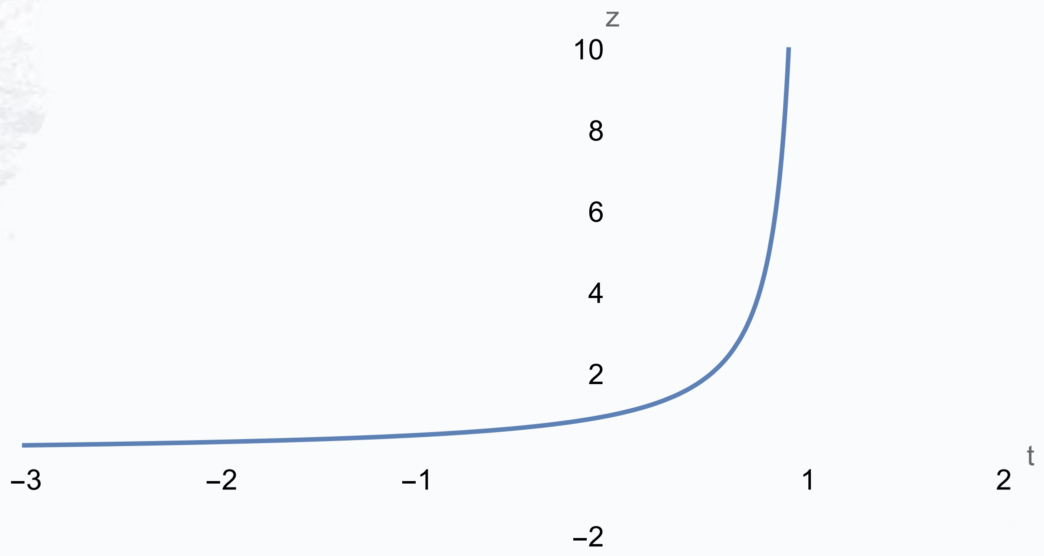
Incompressible Navier-Stokes equation

- Hydrodynamic model of viscous fluids
  - u is the velocity of the fluid
  - p is the pressure
  - $E = \int |u|^2$ is kinetic energy
- Millennium Prize Problem
  - "If u_0 is nice, will the solution blowup?"

$$u_t + (u \cdot \nabla)u + \nabla p = \nu \triangle u$$
$$\nabla \cdot u = 0$$
$$u\Big|_{t=0} = u_0 : \mathbb{R}^3 \to \mathbb{R}^3$$

- Blowup in ordinary differential equations
  - Consider $\frac{dz}{dt} = z^2$
  - If $z(0) = z_0$, this has solution

$$z(t) = \frac{z_0}{1 - z_0 t}$$



Incompressible Navier-Stokes equation

Vorticity formulation

- Viscosity/ Diffusion
- Vortex Stretching
- Convection
- Incompressibility/ Nonlocality

$$\omega_t + (u \cdot \nabla)\omega = \nu \triangle \omega + (\omega \cdot \nabla)u$$

$$\omega = \nabla \times u$$

$$\omega\Big|_{t=0} = \omega_0 : \mathbb{R}^3 \to \mathbb{R}^3$$

Toy Models: Burgers, Fujita, etc

- Let $u(t,x): [0,T) \times \mathbb{R} \to \mathbb{R}$

$$u_t + uu_x = 0$$ Blow-up!

$$u_t + uu_x = u_{xx}$$ No blow-up

- Let $v = u_x$ (or $u = \int v\, dx$)

$$v_t + v^2 = v_{xx}$$ Blow-up!

$$v_t + uv_x + v^2 = v_{xx}$$ No blow-up

$$v_t - uv_x + v^2 = v_{xx}$$ Blow-up!

Viscosity alone is not enough to suppress the blow-up.

But perhaps blow-up can be prevented by viscosity and/or an appropriate nonlinear convection.

Hisashi Okamoto, 2018

"Some Navier-Stokes problems which I cannot solve"

Vortex stretching: $\omega \cdot \nabla u$

- Using $\omega \mapsto H\omega$ to model $\omega = \nabla \times u \mapsto \nabla u$ Constantin-Lax-Majda (1985) proposed the inviscid 1D equation

$$\partial_t \omega = \omega\, H(\omega)$$

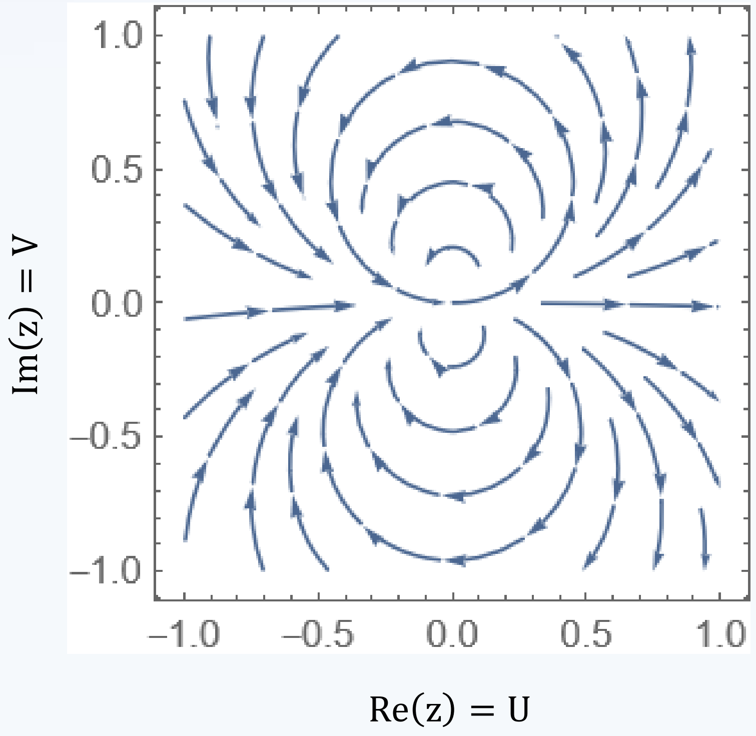
- The Hilbert transform
  - $H(\omega)(x) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{\omega(y)}{x-y} dy$
  - Skew-symmetric: $H^2 = -Id$
- For $z = H\omega + i\omega$ we obtain complex diff. eq.

$$\partial_t z = \frac{1}{2} z^2$$

  - Blowup $\Leftrightarrow$ $z(x) \in (0, +\infty)$ for any x

For $z = U + i\,V$, this yields the real ODE:

$$2\dot{U} = U^2 - V^2$$
$$2\dot{V} = 2\,UV$$



Constantin-Lax-Majda type models

- To incorporate convection and dissipation, de Gregorio (1990), proposed the following model

$$\omega_t + v\omega_x = \epsilon\, \omega_{xx} + \omega v_x$$
$$v_x = H\omega$$

- Model studied (and modified) by many mathematicians

- Neither convection nor dissipation alone is sufficient to prevent blowup!

For $z = H\omega + i\omega$, the CLM equation can be written as $z_t = \frac{1}{2} z^2$

A Toy Model: For $u: \mathbb{T} \to \mathbb{C}$, consider

$$u_t = e^{i\phi}(u_{xx} + u^2)$$

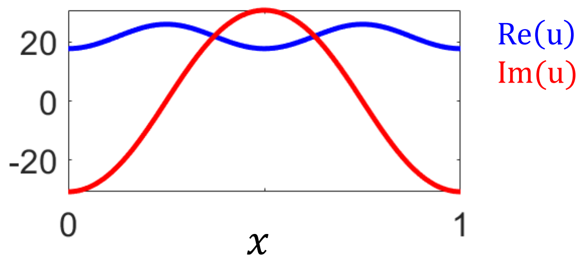
ϕ	Type	Fluid
0	Heat	High Viscosity
$\pi/4$	Complex Ginzberg Landau	Med. Viscosity
$\pi/2$	Nonlinear Schrodinger Eq	No Viscosity

Outline

- Part 1: Introduction
- Part 2: What is a computer assisted proof?
- Part 3: A toy model for fluid dynamics
- **Part 4: Global dynamics and blowup**

Global dynamics of $u_t = i(\Delta u + u^2)$

JJ, Lessard, Takayasu; *Adv. Math* (2022)

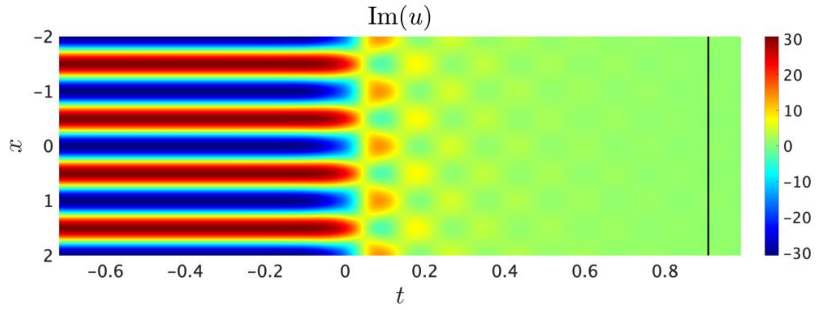


Nontrivial Equilibrium

Cartoon phase space of ∞-dimensional PDE dynamics

Global dynamics of $u_t = i(\triangle u + u^2)$

JJ, Lessard, Takayasu; *Adv. Math* (2022)



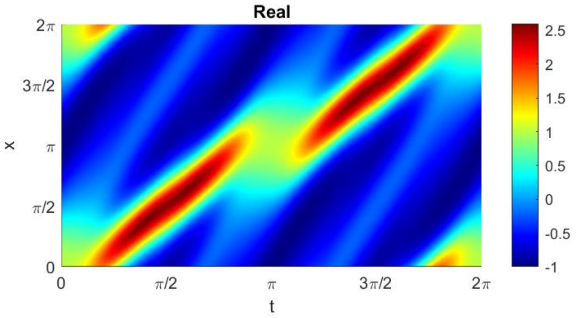
Heteroclinic Orbit

Cartoon phase space of ∞-dimensional PDE dynamics

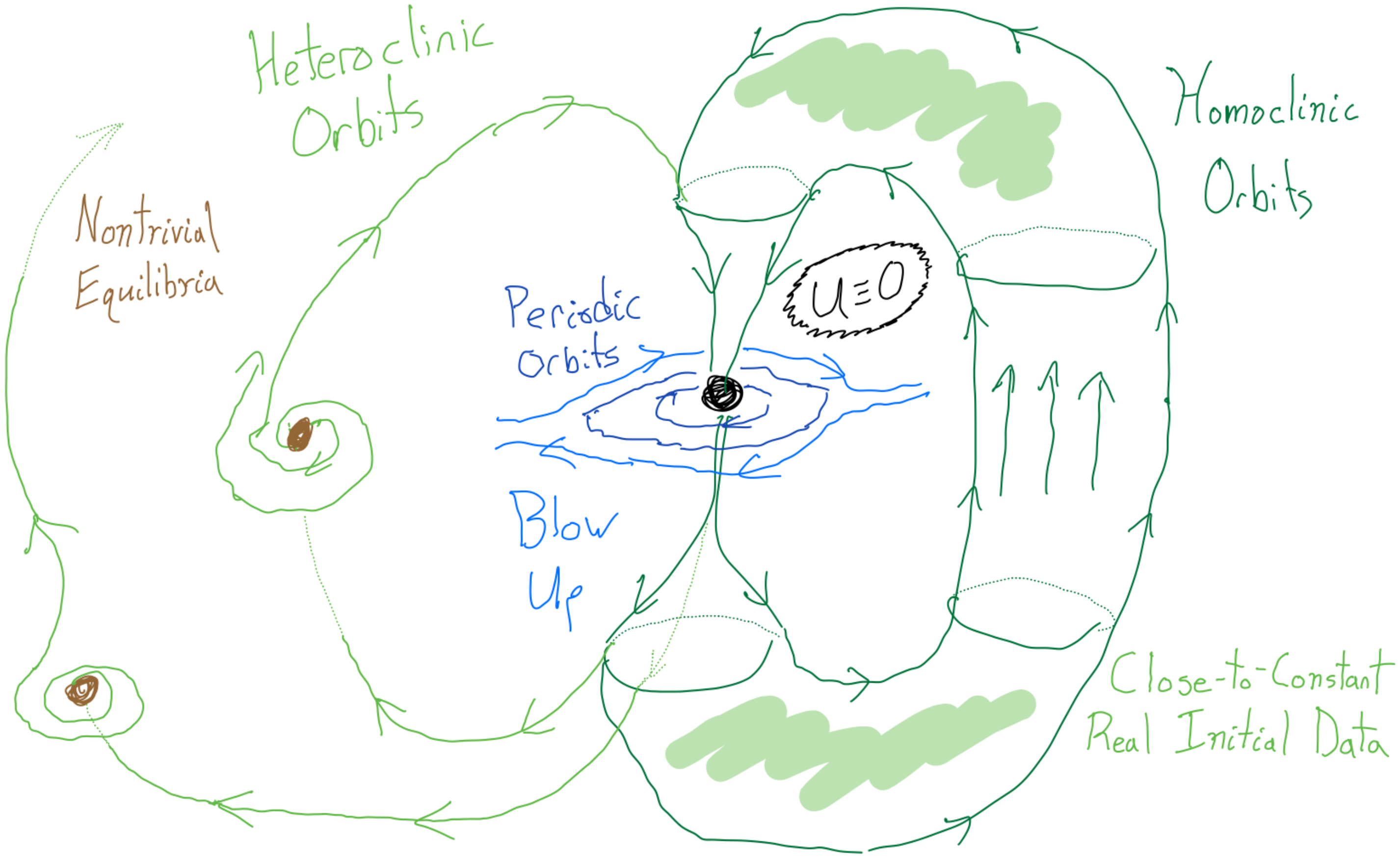
Global dynamics of $u_t = i(\triangle u + u^2)$

JJ, Lessard, Takayasu; *Adv. Math* (2022)

JJ; *J. Dynam. Differential Equations* (2022)



Periodic Orbit



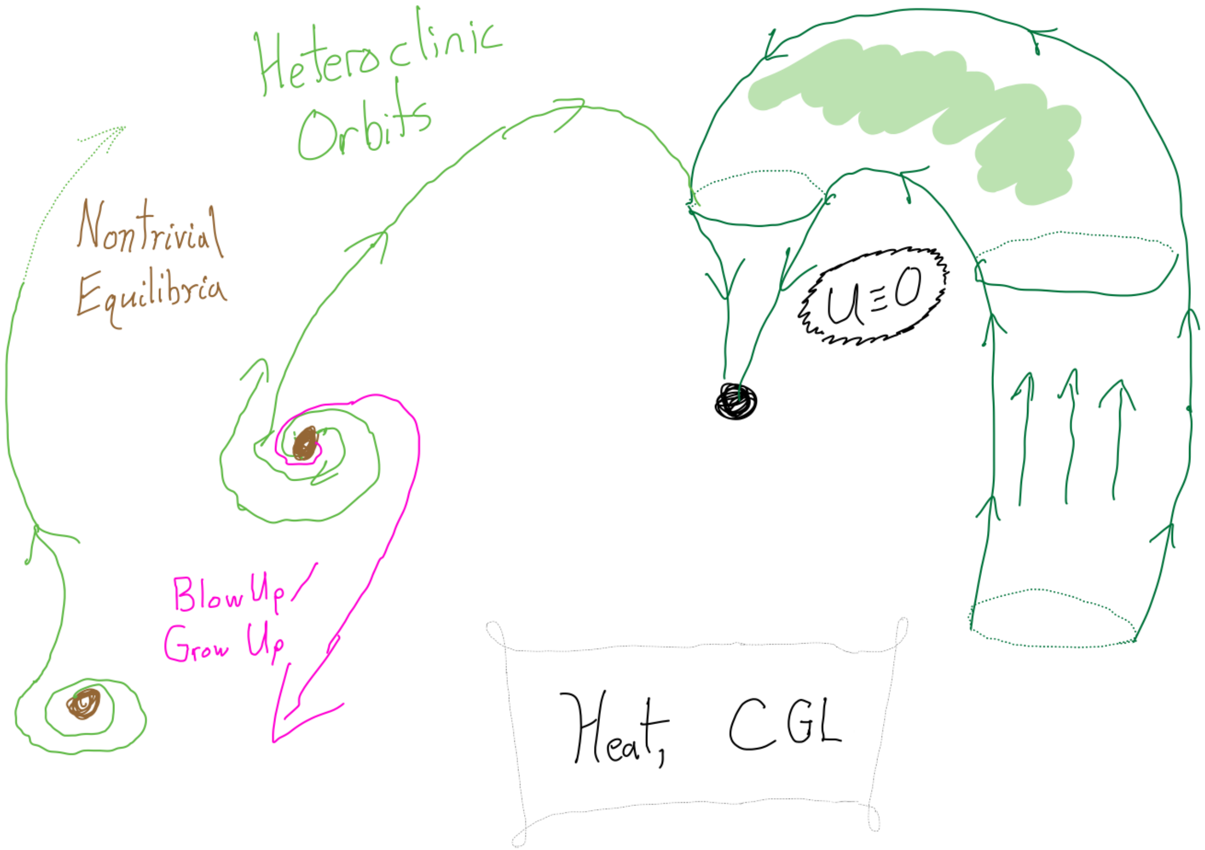
Cartoon phase space of ∞-dimensional PDE dynamics

Global dynamics of $u_t = e^{i\phi}(\triangle u + u^2)$

JJ, Lessard, Takayasu; *Adv. Math* (2022)

JJ; *J. Dynam. Differential Equations* (2022)

JJ, Lessard, Takayasu; *Commun. Nonlinear Sci. Numer. Simul.* (2022)



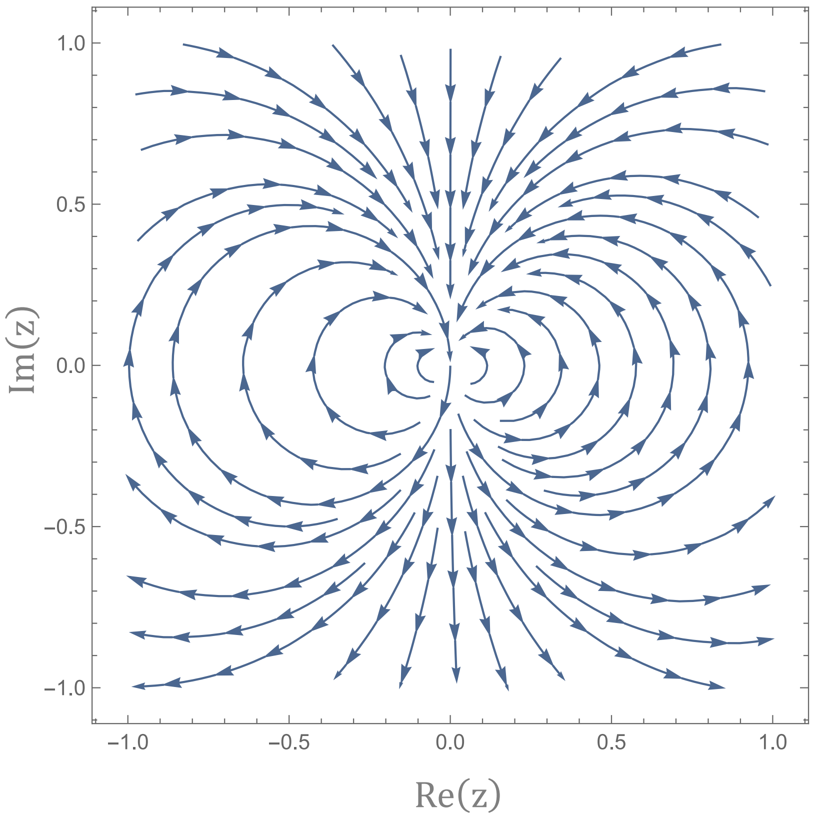
Cartoon phase space of ∞-dimensional PDE dynamics

$\phi = 0, \frac{\pi}{4}$

The NLS $u_t = i(\triangle u + u^2)$ is non-conservative

- **Theorem:** There exists an open set of *homoclinics orbits* (converging to 0 in forward & backward time)
- **Corollary:** Any analytic conserved quantity must be constant
  - If F is continuous and conserved, then $F\big(u(t)\big) = F\big(\lim_{t\to\pm\infty} u(t)\big) = F(0)$
  - $F(u_0)$ must be constant on the open set of homoclinics
  - Constant on open set $\Rightarrow$ globally constant for analytic functionals

Spatially constant dynamics $\dot{z} = iz^2$

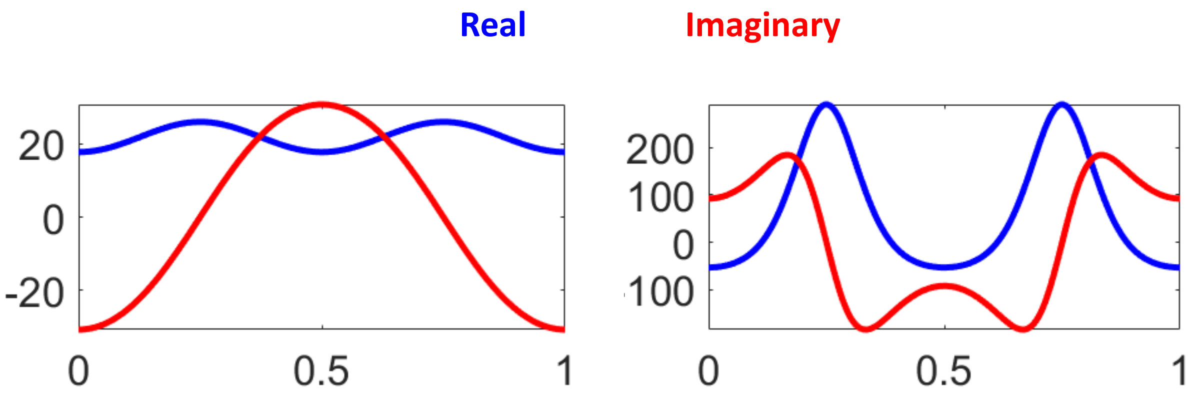


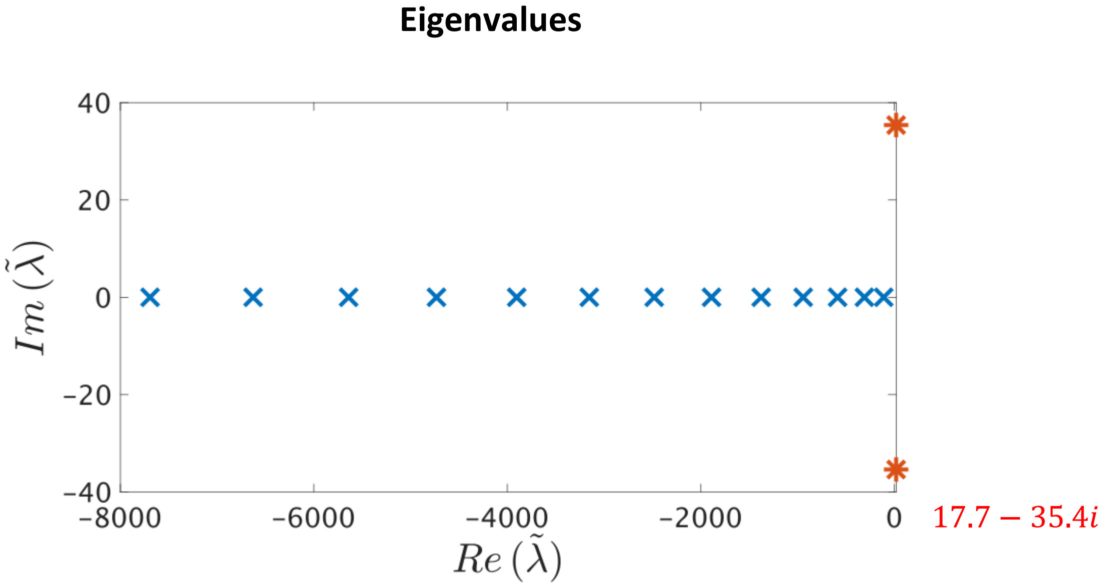
JJ, Lessard, Takayasu; *Adv. Math* (2022)

$$u_t = e^{i\phi}(u_{xx} + u^2)$$

- At least two families of equilibria
  - Homogeneous nonlinearity
  - If $u(t,x)$ is a solution then $n^2 u(n^2 t, nx)$ is a solution
- Computer Assisted Proof
  - Cast as a $F(x) = 0$ problem in Fourier space
  - Use Newton-Kantorovich method
- Linearization about $\tilde{u}$ is unstable

$$e^{i\phi}(h_{xx} + 2\tilde{u}h) = \lambda h$$





Heat $\phi = 0$

JJ, Lessard, Takayasu; *Adv. Math* (2022)

$$u_t = e^{i\phi}(u_{xx} + u^2)$$

- At least two families of equilibria
  - Homogeneous nonlinearity
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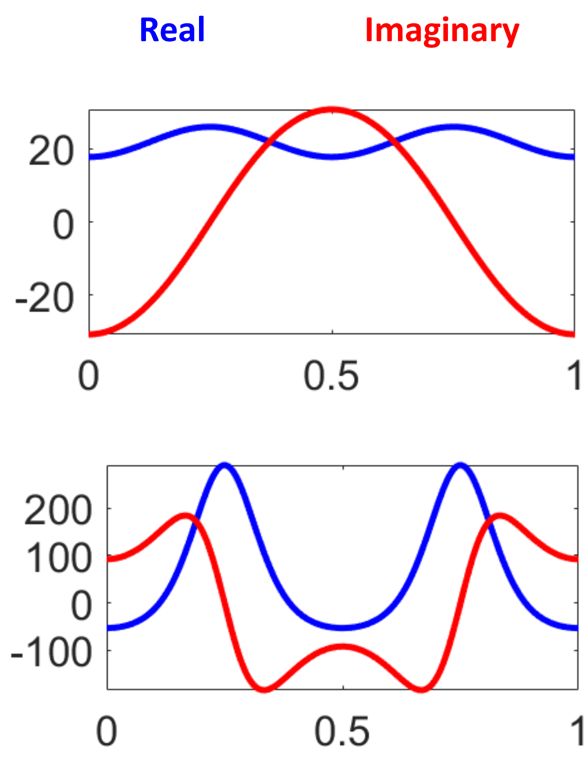
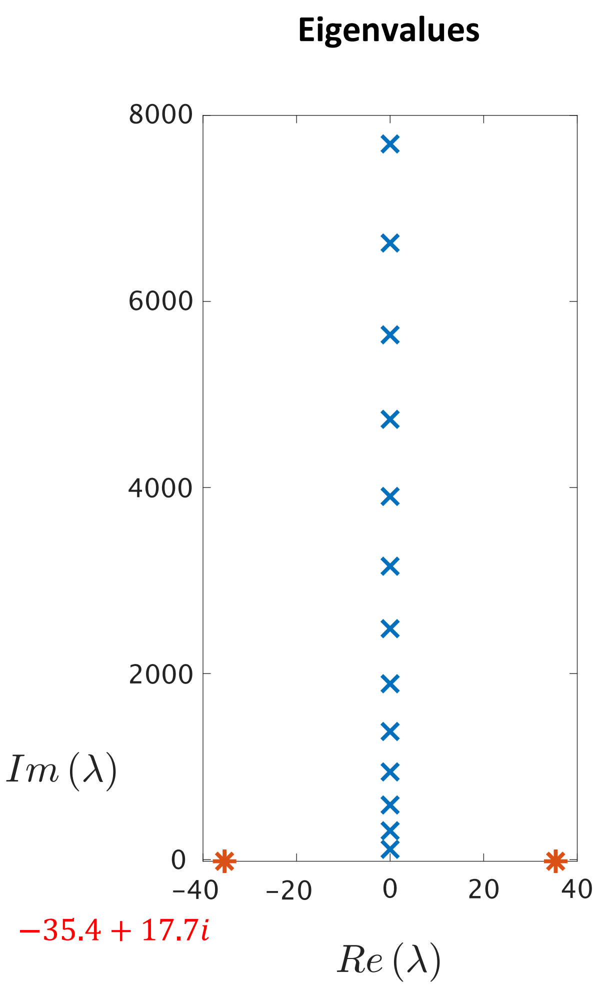




NLS $\phi = \frac{\pi}{2}$

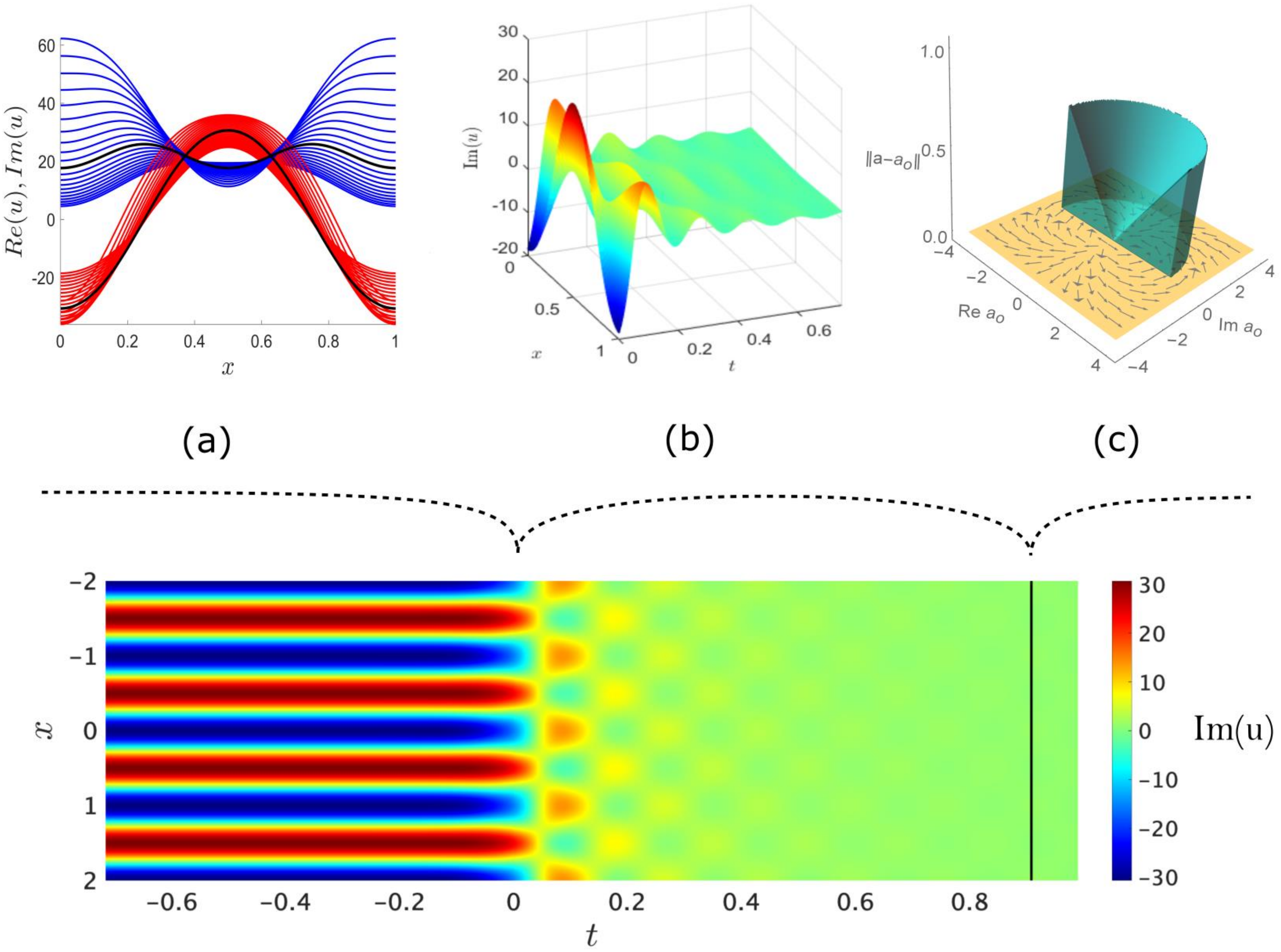
JJ, Lessard, Takayasu; *Adv. Math* (2022)

Computer Assisted Proof of Heteroclinic Orbits

a) Parameterization of unstable manifold

b) Validated integration of the initial value problem

c) Explicit trapping region of solutions converging to the 0 solution



JJ, Lessard, Takayasu; *Adv. Math* (2022)

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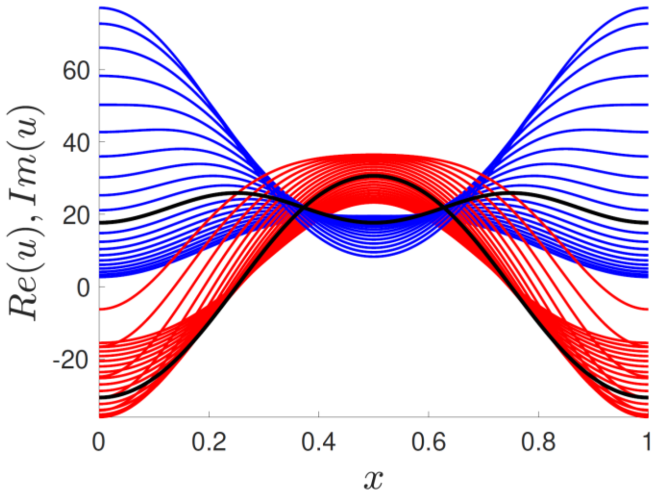

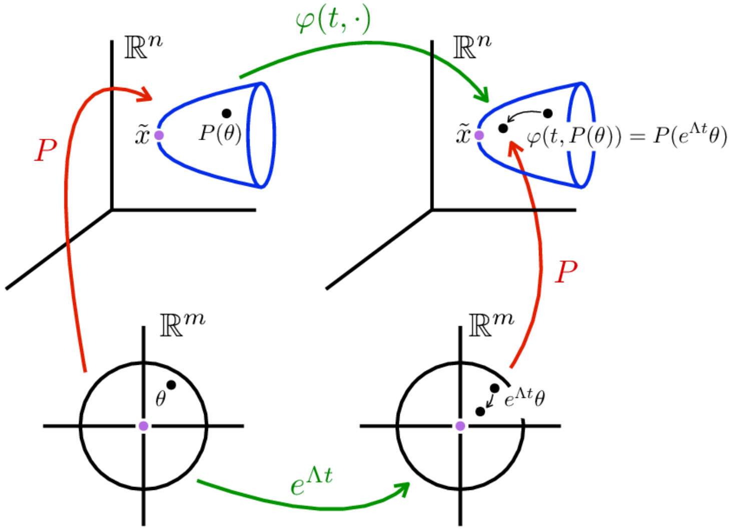

- Look for a chart $P\colon \mathbb{D} \to W^u_{loc}(\tilde{x})$ such that

$$P(0) = \tilde{x}; \quad DP(0) = \xi; \quad \varphi\big(t, P(\theta)\big) = P\big(e^{\lambda t}\theta\big)$$

- Write P as a power series:

$$P(\theta) = \sum_{n=0}^{\infty} p_n\, \theta^n, \qquad p_n \in X$$

- Solve for p_n order-by-order using the parameterization method

Cabré, Fontich & de la Llave, 2003
Reinhardt, & Mireles James, 2019
JJ, Lessard, Takayasu, 2022

Computer Assisted Proof of Heteroclinic Orbits

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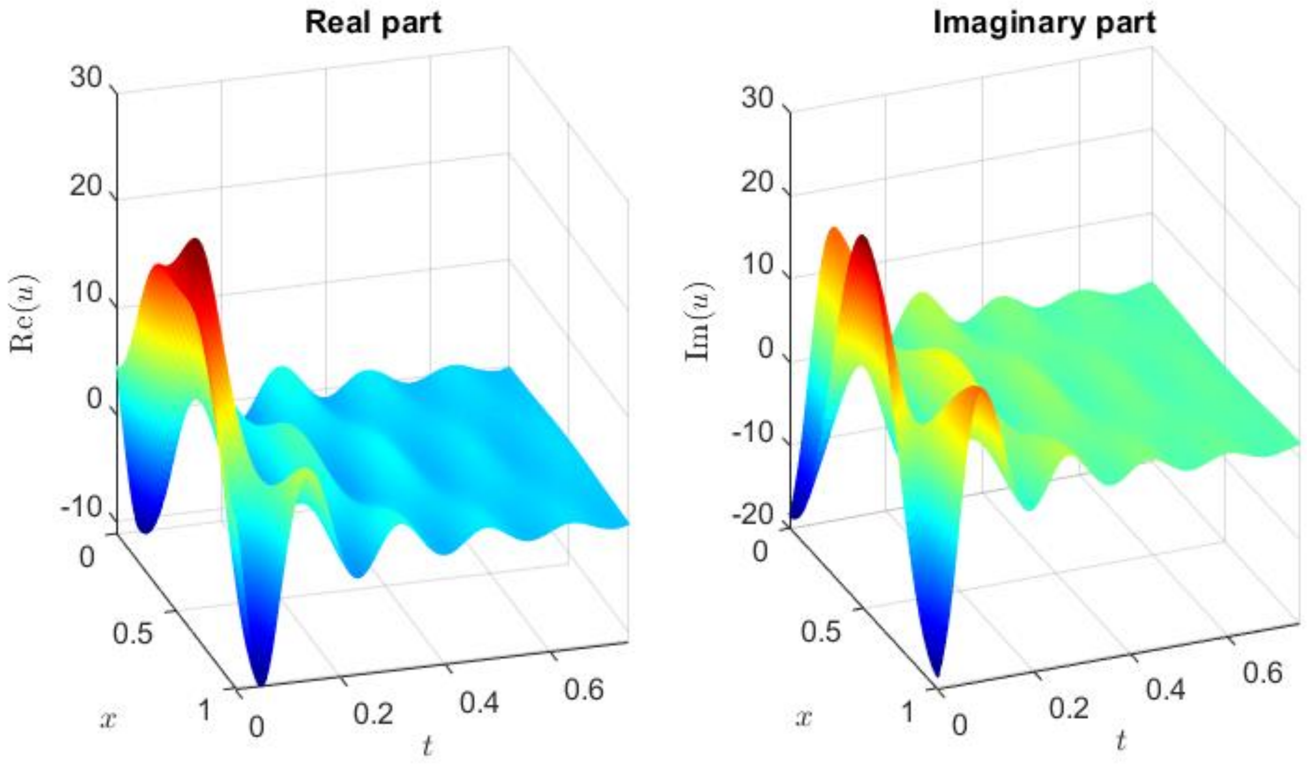

- C_0-semigroup approach to validated integration
  - Compute approximate solution $\tilde{a}(t)$ to IVP
  - Solve linearized problem about $\tilde{a}(t)$
  - Show Picard-like operator is a contraction
  - Propagate errors

Takayasu, et al., 2022
JJ, Lessard, Takayasu, 2022

Computer Assisted Proof of Heteroclinic Orbits

a) Parameterization of unstable manifold

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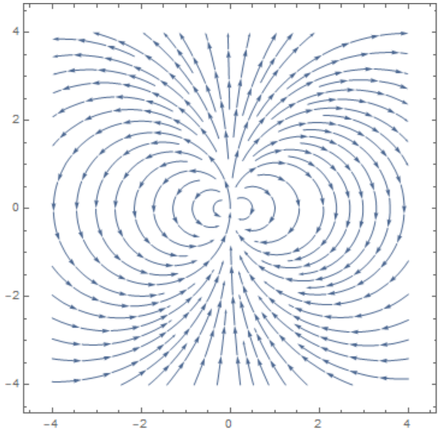
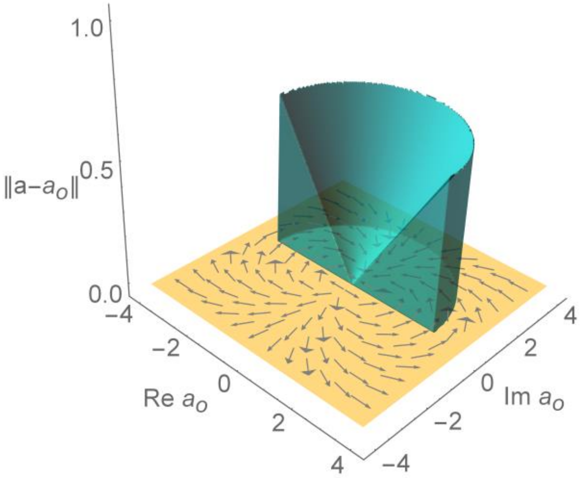
Spatially Constant Dynamics



$iz_t = z^2$



- Center dynamics of the 0-equilibrium
  - Spatially constant solutions have explicit solution $z(t) \sim \mathcal{O}(t^{-1})$
- Blowup coordinates about $z(t)$
  - Make ansatz:

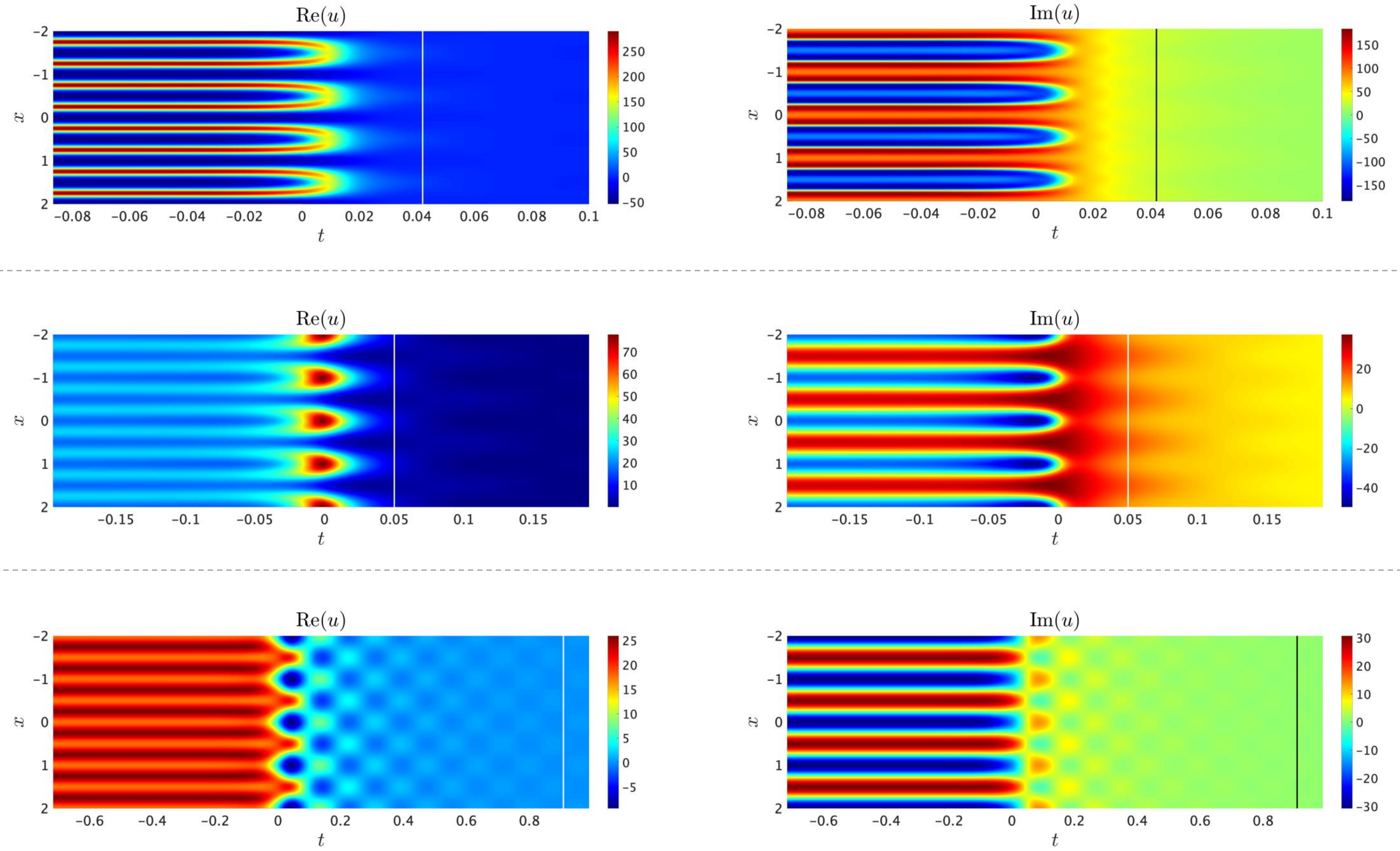
    $$u(t) = z(t) + z(t)^2 \tilde{u}(t)$$

  - The $\tilde{u}(t)$ equation becomes:

    $$i\tilde{u}_t = \tilde{u}_{xx} + z(t)^2 \tilde{u}^2$$

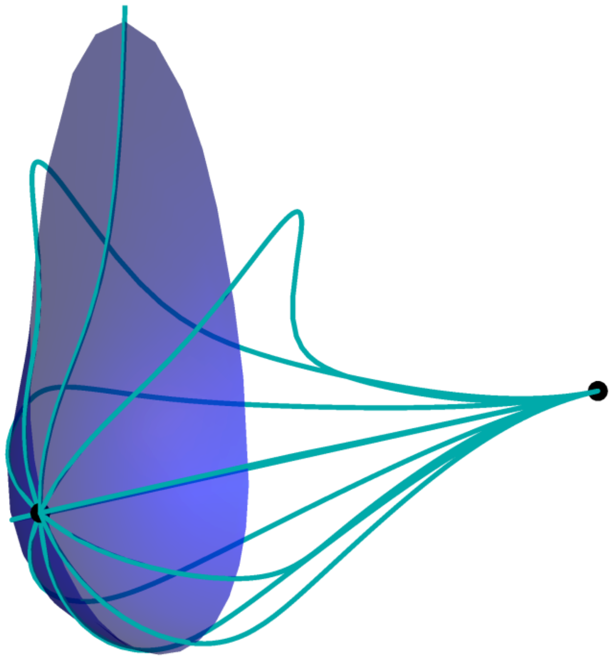
  - Suffices to show $\tilde{u}(t)$ is bounded

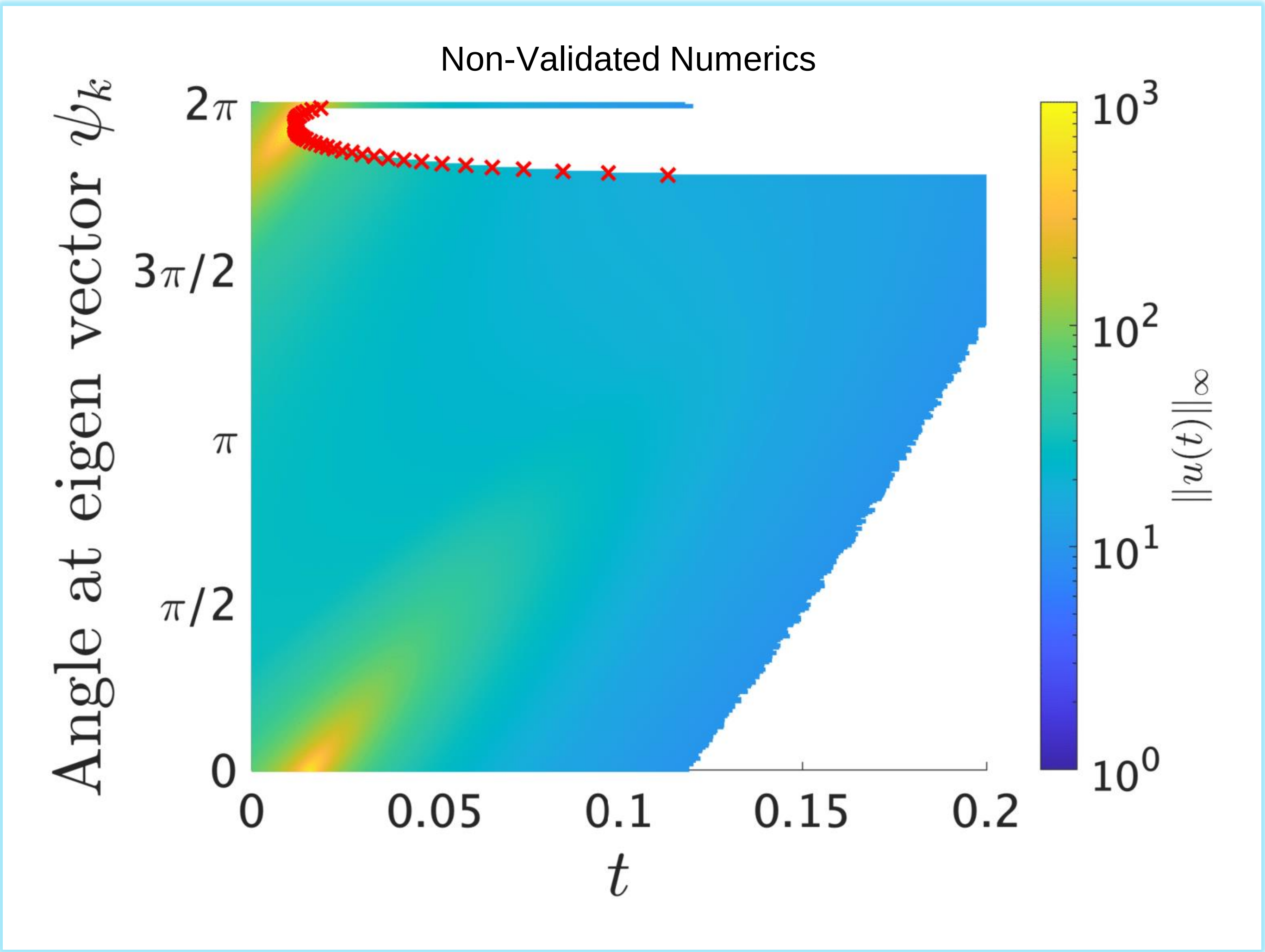
JJ, Lessard, Takayasu, 2022



Re(u)
Im(u)
x
t
-2
-1
0
1
2
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1
250
200
150
100
50
0
-50
150
100
50
0
-50
-100
-150
Re(u)
Im(u)
x
t
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
70
60
50
40
30
20
10
20
0
-20
-40
Re(u)
Im(u)
x
t
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
25
20
15
10
5
0
-5
30
20
10
0
-10
-20
-30

- The (strong) unstable manifold has $\mathbb{C}$ dim. 1
  - Shoot out of different angles $\psi \in \mathbb{S}^1$





JJ, Lessard, Takayasu, *CNSNS* (2022)

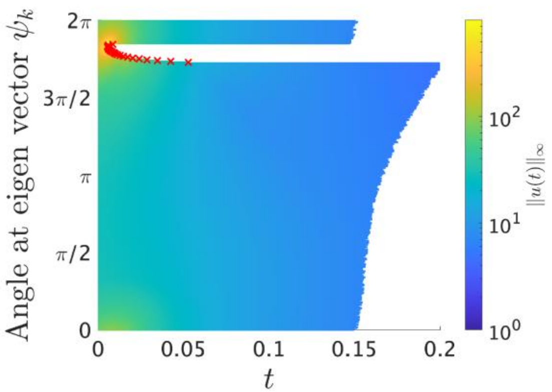
Figure for $\phi = 0$ (Heat) eq.

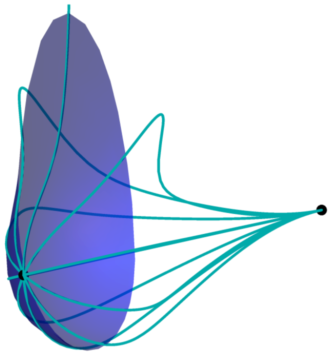
x – inconclusive / C.A.P. failed

no x – C.A.P. of heteroclinic!

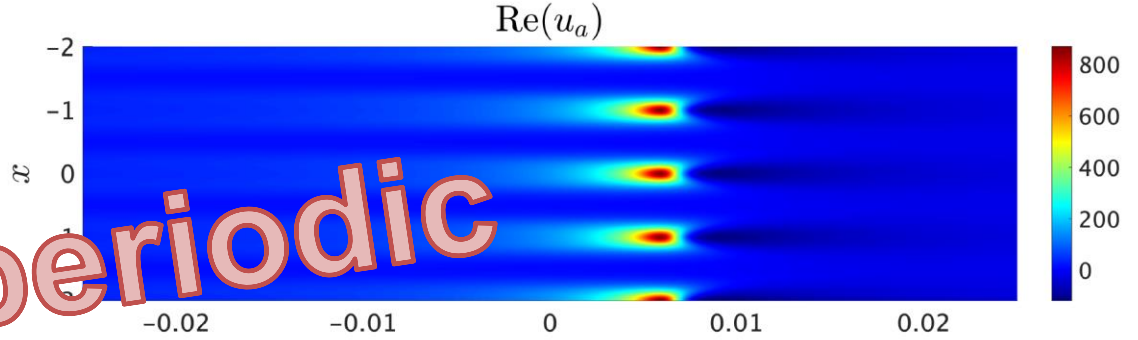
$$u_t = e^{i\phi}(u_{xx} + u^2)$$

- For $\phi \in \{0, \pi/4, \pi/2\}$ we have computer assisted proofs of many connecting orbits
- **Theorem:** Let $\phi \in \left\{0, \frac{\pi}{4}\right\}$
  - The unstable manifold of the nontrivial equilibrium contains an unbounded trajectory



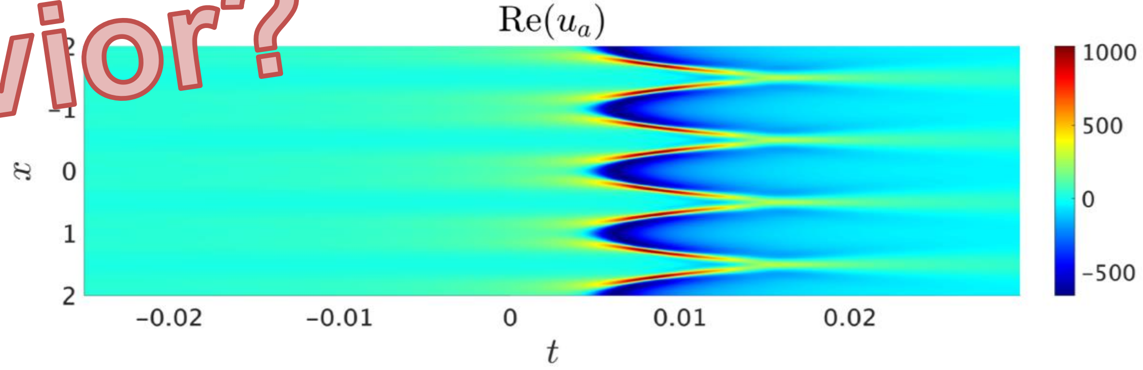


$\psi = 5.81$
CAP



Is there periodic behavior?

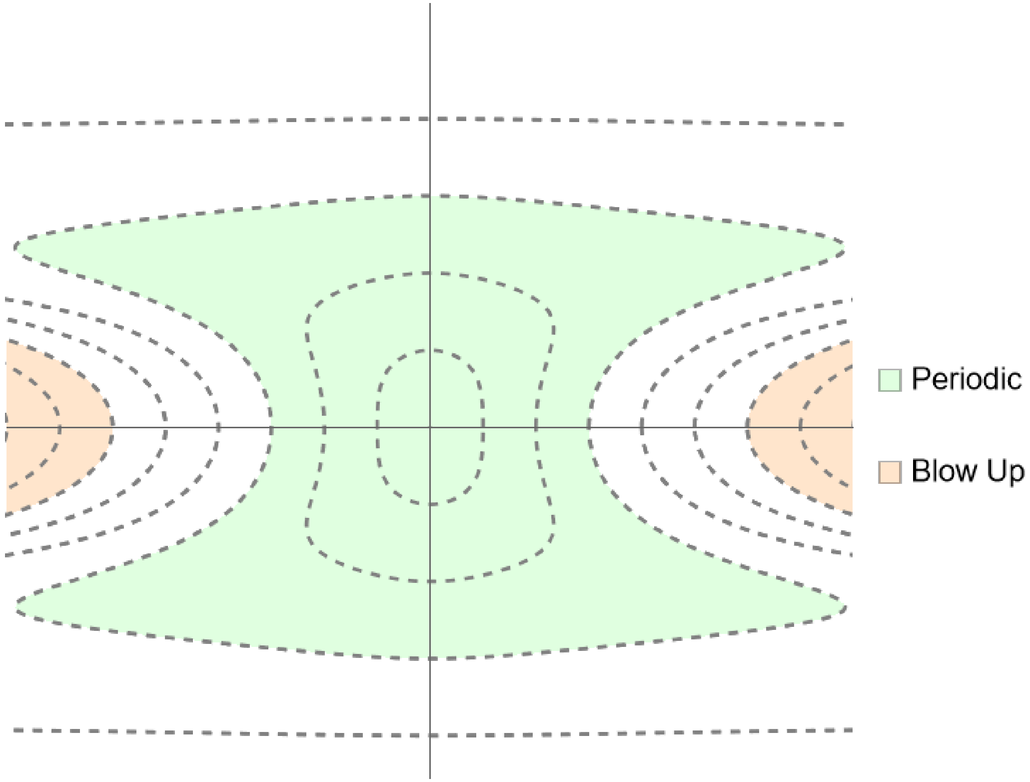
$\psi = 5.71$
no CAP



Figures for $\phi = \pi/4$ (CGL) eq.

JJ, Lessard, Takayasu, *CNSNS* (2022)

Theorem: The space of positive Fourier modes of the PDE $iu_t = \triangle u + u^2$ on $\mathbb{T}^d$ has two types of solutions: periodic and blowup



Cartoon family of periodic solutions limiting to blowup solutions

Theorem: Fix initial data $u_0(x) = \sum_{n\in\mathbb{N}_*^d} \gamma_n \, e^{inx}$

- The solution is given as $$u(t,x) = \sum_{n\in\mathbb{N}_*^d} a_n(t)\, e^{inx}$$ where the functions a_n are 2π periodic, and recursively defined
- If $\sum_{n\in\mathbb{N}_*^d} |\gamma_n| < \frac{1}{4}$, then $u(t)$ is bounded and 2π periodic

JJ; *J. Dynam. Differential Equations* (2022)

Theorem: The space of positive Fourier modes of the PDE $iu_t = \triangle u + u^2$ on $\mathbb{T}^d$ has two types of solutions: periodic and blowup

If $d = 1$ and $a_k = 0 \ \forall \ k \leq 0$, then

$$\dot{a}_1 = i\omega^2 a_1$$

$$\dot{a}_2 = i\omega^2 2^2 a_2 - i a_1^2$$

$$\dot{a}_3 = i\omega^2 3^2 a_3 - 2i a_1 a_2$$

$$\dot{a}_4 = i\omega^2 4^2 a_4 - i(2a_1 a_3 + a_2^2)$$

$$\vdots$$

If we take monochromatic initial data $u_0(x) = A\, e^{i\omega x}$ *then ...*

- $a_1(t) = Ae^{i\omega^2 t}$
- $a_2(t) = \frac{A^2}{\omega^2}\left(\frac{e^{2i\omega^2 t}}{2} - \frac{e^{4i\omega^2 t}}{2}\right)$
- $a_3(t) = \frac{A^3}{\omega^4}\left(\frac{e^{3i\omega^2 t}}{6} - \frac{e^{5i\omega^2 t}}{4} + \frac{e^{9i\omega^2 t}}{12}\right)$
- $a_4(t) = \frac{A^4}{\omega^6}\left(\frac{7e^{4i\omega^2 t}}{144} - \frac{e^{6i\omega^2 t}}{10} + \frac{e^{8i\omega^2 t}}{22} + \frac{e^{10i\omega^2 t}}{36} - \frac{11e^{16i\omega^2 t}}{1440}\right)$

JJ; *J. Dynam. Differential Equations* (2022)

Theorem: The space of positive Fourier modes of the PDE $iu_t = \triangle u + u^2$ on $\mathbb{T}^d$ has two types of solutions: periodic and blowup

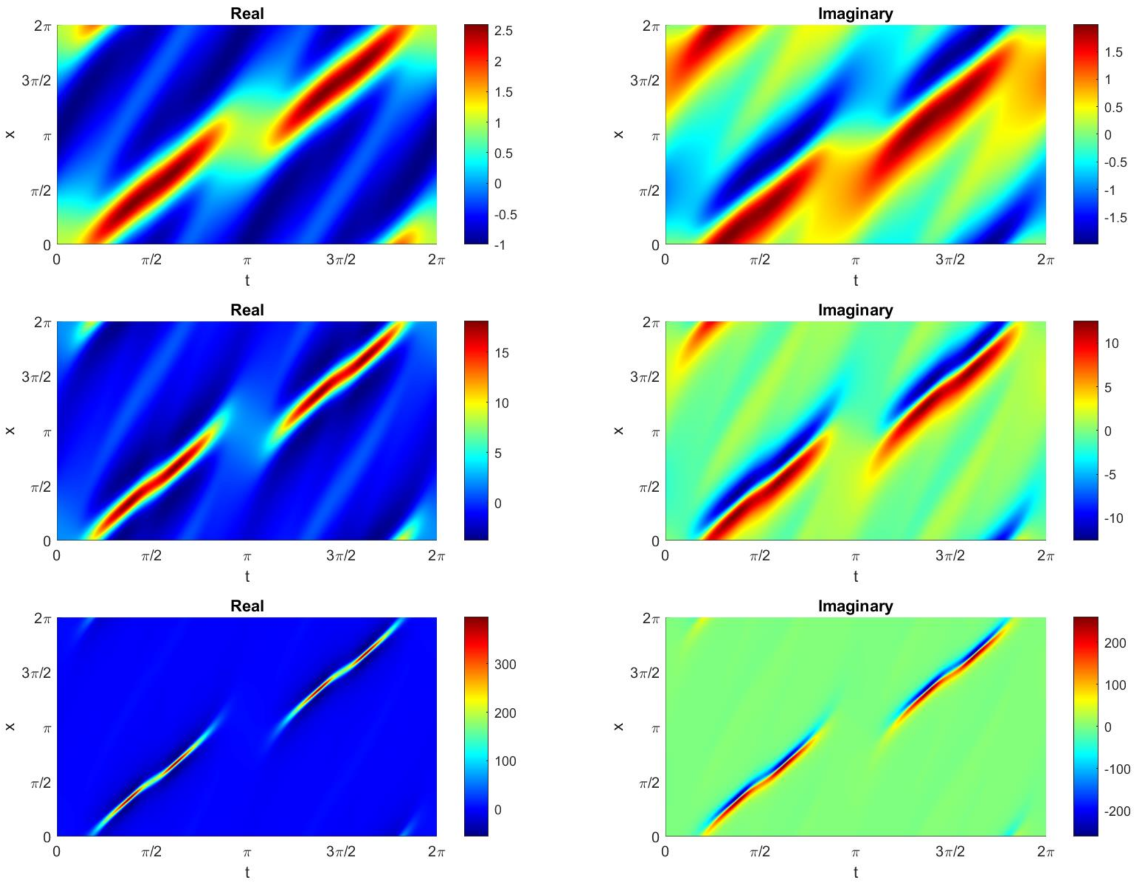
Theorem: Consider the initial data $u_0(x) = A\, e^{ix}$

- If $|A| \leq 3$ then the solution is 2π periodic
- If $|A| \geq 6$ then the solution blows up in finite time in the L^2 norm, with $T^* < 2\pi$
- The solution exists for all time (and is periodic) if and only if $|A| < A^*$

$A = 1$

$A = 2$

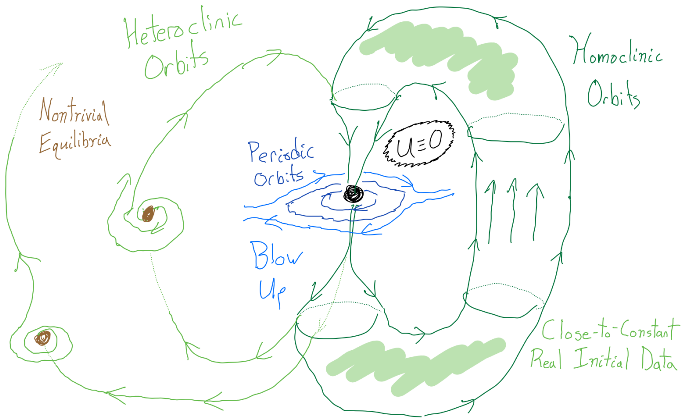
$A = 3$



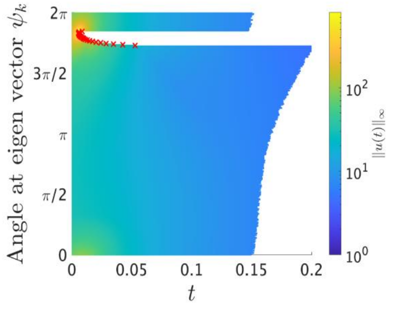
JJ; *J. Dynam. Differential Equations* (2022)

Conclusions

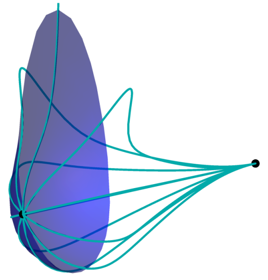
- **Summary**
  - Found new dynamics in
    $$u_t = e^{i\phi}(\triangle u + u^2)$$
    - Equilibria, connecting orbits, periodic orbits, blowup-solutions
  - Developed new methodologies
- **Take-home message**
  - Found singularities by following the dynamics
  - Computer assisted proofs provide a canary in the coal mine



Cartoon phase space of ∞-dimensional PDE dynamics



(Left) Norm of solutions exiting the equilibrium's unstable manifold



(Right) Cartoon drawing of unstable manifold

References & Related Work

- Chen, J, and Hou T. (2022) "Stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth data." *arXiv preprint* arXiv:2210.07191.
- Cho, C. H., Okamoto, H., & Shōji, M. (2016). A blow-up problem for a nonlinear heat equation in the complex plane of time. *Japan Journal of Industrial and Applied Mathematics*, 33(1), 145-166.
- Constantin, P., Lax, P. D., & Majda, A. (1985). A simple one-dimensional model for the three-dimensional vorticity equation. *Communications on pure and applied mathematics*, 38(6), 715-724.
- Curry, J. H., Herring, J. R., Loncaric, J., & Orszag, S. A. (1984). Order and disorder in two-and three-dimensional Bénard convection. *Journal of Fluid Mechanics*, 147, 1-38.
- De Gregorio, S. (1990). On a one-dimensional model for the three-dimensional vorticity equation. *Journal of statistical physics*, 59(5), 1251-1263.
- **JJ** (2022). Quasiperiodicity and blowup in integrable subsystems of nonconservative nonlinear Schrödinger equations. *Journal of Dynamics and Differential Equations*, 1-25.
- **JJ**, Lessard, JP., and Takayasu, A. (2022a) Global dynamics in nonconservative nonlinear Schrödinger equations. *Advances in Mathematics*, 398, 108234.
- **JJ**, Lessard, JP., and Takayasu, A. (2022b) Singularities and heteroclinic connections in complex-valued evolutionary equations with a quadratic nonlinearity. *Communications in Nonlinear Science and Numerical Simulation*, 107, 106188.
- Okamoto, H., Sakajo, T., & Wunsch, M. (2008). On a generalization of the Constantin–Lax–Majda equation. *Nonlinearity*, 21(10), 2447.
- Takayasu, A., Lessard, JP., **JJ**, and Okamoto H. (2022) Rigorous numerics for nonlinear heat equations in the complex plane of time. *Numerische Mathematik,* 1-58.